Math 724, Fall 2017
Homework #3
Deadline: Monday, October 2, 5:00pm

Instructions: Typeset your solutions in LaTeX. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name and the problem set number (e.g., `Euler3.pdf`). Collaboration is encouraged, but each student must write up his or her solutions independently and acknowledge all collaborators.

Problem #1 Bogart, Problem #82. (I know we did this in class, but I would like you to produce a careful writeup.)

Problem #2 Bogart, Problem #83.

Problem #3 Bogart, Problem #116. (Ditto. If necessary, review problems #113–115 first. If you didn't get to them in class, do them now; they should be relatively straightforward once you understand how the Prüfer code works.)

Problem #4 Let $G = (V, E)$ be a simple graph with n vertices ($n \geq 2$). ("Simple" means that every edge has two different vertices as its endpoints, and no two edges have the same pair of endpoints. Therefore, the degree of a vertex is its number of neighbors.) Prove that some two vertices of G have the same degree.

Problem #5 Let $G = (V, E)$ be a simple graph. Prove that the following conditions are equivalent. (Some of the implications should follow from problems done in class, but you should still give complete proofs. I.e., don't cite problems from Bogart in your answers.)

(1) G is connected and acyclic.
(2) G is connected and $|E| = |V| - 1$.
(3) G is acyclic and $|E| = |V| - 1$.
(4) Every two vertices in G are joined by exactly one path.

Problem #6 Let $G = (V, E)$ be a connected simple graph, with vertex set $V = \{v_1, \ldots, v_n\}$. The *Laplacian matrix* of G is the $n \times n$ square matrix $L(G) = [\ell_{ij}]_{i,j=1}^n$ with entries

$$\ell_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

The *reduced Laplacian* $L^i(G)$ is the $(n-1) \times (n-1)$ matrix obtained from $L(G)$ by crossing out the i^{th} row and column.

(a) Prove that for a given graph G, the number $\det L^i(G)$ is independent of the choice of i. (This is really a matter of linear algebra rather than combinatorics.) Henceforth, call that number $\tau(G)$.
(b) Calculate $\tau(G)$ for several different graphs, including trees, cycles, and complete graphs. Feel free to do this with a computer system such as Sage. What do you notice? Can you prove your conjecture in any special cases?

Here is Sage code to compute $\tau(K_4)$. You can copy and paste it into the Sage cell server and click the Evaluate button. Then change the second and third lines to play with different graphs.

```
G = graphs.CompleteGraph(4)
L = G.laplacian_matrix()
LRed = L.matrix_from_rows_and_columns([0,1,2],[0,1,2])
det(LRed)
```

Some notes:

- Sage knows how to construct several "standard" graphs. To see a list of them, type `G = graphs.` (including the period!) and then hit the Tab key.
- Sage numbers the rows and columns of lists, matrices, etc. starting at 0. So the rows and columns of `L` are numbered 0,1,2,3, and the third line of code above defines the reduced Laplacian `LRed` by restricting to rows and columns 0,1,2, i.e., deleting rows and column 3.