Math 724, Fall 2013
Problem #211(c)

Let $y = \sum_{i=0}^{\infty} a_i x^i$. Taking the original recurrence, multiplying both sides by x^i¡ and summing over i gives

$$\sum_{i=1}^{\infty} a_i x^i = \sum_{i=1}^{\infty} 3a_{i-1}x^i + \sum_{i=1}^{\infty} 2^i x^i.$$

Therefore

$$\begin{aligned} y - a_0 &= \sum_{j=0}^{\infty} 3a_j x^{j+1} + \sum_{j=0}^{\infty} 2^{j+1} x^{j+1} \\ &= 3x \sum_{j=0}^{\infty} a_j x^j + 2x \sum_{j=0}^{\infty} 2^j x^j \\ &= 3xy + \frac{2x}{1-2x}. \end{aligned}$$

Solving this equation for y gives

$$y = \underbrace{\frac{a_0}{1-3x}}_{A} + \underbrace{\frac{2x}{(1-2x)(1-3x)}}_{B}.$$

First, let's take care of the easy part:

$$A = a_0 \sum_{i=0}^{\infty} (3x)^i \ = \ \sum_{i=0}^{\infty} 3^i a_0 x^i.$$

Second, let's work on B:

$$B = \left(\sum_{p=1}^{\infty} 2^p x^p \right) \left(\sum_{q=0}^{\infty} 3^q x^q \right).$$

When we multiply out the two infinite sums, the contributions to the coefficient of x^i will come from pairs of summands with $p + q = i$, i.e., $q = i - p$. That is:

$$\begin{aligned} B &= \left(\sum_{p=1}^{\infty} 2^p x^p \right) \left(\sum_{q=0}^{\infty} 3^q x^q \right) \\ &= \sum_{i=0}^{\infty} \left(\sum_{p=1}^{i} 2^p 3^{i-p} \right) x^i \\ &= 2 \cdot 3^{i-1} \sum_{i=0}^{\infty} \left(\sum_{p=0}^{i-1} (2/3)^p \right) x^i \qquad (1) \end{aligned}$$

This sum is a partial geometric series and so we can boil it down. Specifically,

$$\frac{1-z^i}{1-z} \ = \ z^{i-1} + z^{i-2} + \cdots + z + 1 \ = \ \sum_{p=0}^{i-1} z^p$$

so we can rewrite (1) as

$$\begin{aligned} B &= 2 \cdot 3^{i-1} \sum_{i=0}^{\infty} \left(\frac{1-(2/3)^i}{1-(2/3)} \right) x^i \\ &= 2 \cdot 3^i \sum_{i=0}^{\infty} (1-(2/3)^i) x^i \\ &= 2 \sum_{i=0}^{\infty} (3^i - 2^i) x^i \end{aligned}$$

Putting everything back together, we get

$$y = A + B = \sum_{i=0}^{\infty} \left[3^i a_0 + 2(3^i - 2^i) \right] x^i$$

so

$$\boxed{\boldsymbol{a_i = 3^i a_0 + 2(3^i - 2^i).}}$$

Let's check this in Sage:

```
var('a0')
def recursiveA(i):
    if i == 0:
        answer = a0
    else:
        answer = 3 * recursiveA(i-1) + 2^i
    return answer
closedA = lambda i: 3^i * a0 + 2 * 3^i * (1-(2/3)^i)

[recursiveA(i) - closedA(i) for i in range(10)]
## Look Ma, all zeros!
```

Here is a better way to attack B, suggested by Kyle in class on 11/8: expand it in partial fractions. Write

$$B = \frac{2x}{(1-2x)(1-3x)} = \frac{P}{1-2x} + \frac{Q}{1-3x}.$$

Cross-multiplying and clearing denominators gives

$$2x = P(1-3x) + Q(1-2x) = (P+Q) + (-3P-2Q)x.$$

Therefore $P + Q = 0$ and $-3P - 2Q = 2$. Solving this system gives $P = -2$ and $Q = 2$, so

$$B = -\frac{2}{1-2x} + \frac{2}{1-3x} = \sum_{i=0}^{\infty} 2(3^i - 2^i) x^i$$

with less hassle than before.