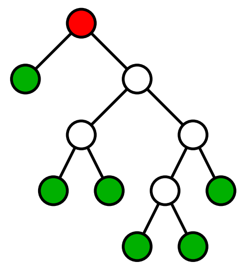
Math 724, Fall 2013
Extra Problems for §1.3.1

Problem 52A: Start with a convex polygon P with n sides ($n \geq 3$). A chord of P is a line segment that joins two of its vertices. By drawing $n-3$ chords, you can subdivide P into $n-2$ triangles. Note that not every choice of $n-3$ chords will work, because two chords that cross each other cannot both be used.

In terms of n, how many ways are there to subdivide P into triangles? Prove your answer bijectively.

Problem 52B: The thing shown in the following figure is a *plane binary tree.*



The red vertex is called the *root*. The green vertices are *leaves*. Every vertex that is not a leaf has two *children*, one on the left and one on the right. Every vertex that is not the root has a unique *parent*. The left-to-right ordering of children matters — for example, if we reflecting the given tree left-to-right, it would *not* be the same tree.

In terms of n, how many plane binary trees are there with exactly n leaves? Prove your answer bijectively.

Problem 52C: Let $p = (p_1, \ldots, p_n)$ be a permutation of the numbers $1, \ldots, n$. We want to look at permutations in which three numbers can only appear in certain **relative order**. First of all, we have to say what relative order means. I could tell you a definition, but I think you'll believe it more if you figure out for yourself what it ought to be based on the following examples:

Permutation	Relative order of underlined digits
$3\underline{2}\underline{4}15\underline{6}$	123
$1\underline{2}\underline{5}\underline{3}46$	132
$1\underline{6}2\underline{4}5\underline{3}$	321
$1\underline{6}24\underline{5}\underline{3}$	321
$1\underline{6}\underline{2}45\underline{3}$	312

(a) Start by writing down a precise definition of "relative order:"

If $p \in S_n$ and $q \in S_m$ are permutations, then p is said to be ***q*-free** if no m digits of p are in the relative order given by q. For example, 4763521 is 123-free, but $\underline{4}73\underline{5}2\underline{6}1$ is not. For $q \in S_3$, let $f_n(q)$ be the number of q-free permutations in S_n.

(b) What can you say about the numbers $f_n(q)$ for different n and q? To get a feel for the problem, write out the explicit lists for $n = 1, 2, 3$ by hand. For $n \geq 4$, you may want to use a computer.