Math 410, Spring 2013
Solution to HW Problem #21

The range of f is $[0,1]$. (Note that it is not $(0,1)$, since $f(0,0)=0$ and $f(1,1)=1$ — for the latter, write 1 as $0.9999\ldots$.) Also, note that the domain of f is the set of ordered pairs of numbers (x,y) in $[0,1]$. You can think of this domain as the closed unit square in $\mathbb{R}^2$; let's give it the name S.

The function f is one-to-one for the following reason. Suppose that (x,y) and (x',y') aren two different elements of the square S; that is, either $x \neq x'$ or $y \neq y'$ (or possibly both). Say $x \neq x'$. Then there has to be some k such that the k^{th} digits of the decimal expansions of x and x' are different (if there were no such k, then x and x' would be the same number, which they aren't). By the construction of f, this means that the $(2k-1)^{st}$ digits in the decimal expansions of $f(x,y)$ and $f(x',y')$ are different. Therefore, $f(x,y) \neq f(x',y')$. A similar argument would work if $y \neq y'$. The upshot is that if $(x,y) \neq (x',y')$, then $f(x,y) \neq f(x',y')$, which is precisely the statement that f is one-to-one.

Meanwhile, the function f is onto for the following reason. Suppose r is any real number in $[0,1]$. Write its decimal expansion as $0.r_1r_2r_3r_4\ldots$. Then we can construct real numbers x,y such that $f(x,y)=r$, namely

$$x = 0.r_1r_3r_5r_7\ldots, \qquad y = 0.r_2r_4r_6r_8\ldots$$

This works for every r, so every $r \in [0,1]$ is of the form $f(x,y)$ for some $(x,y) \in S$, so the function f is onto.

We have now shown that f is a bijection. Therefore, its domain and range have the same cardinality. That is, what we have proved is that **a line segment contains exactly as many points as a square**. This may seem counterintuitive, since the segment is a 1-dimensional object with no area, and the square is a 2-dimensional object — but we have constructed a bijection. This is another of those instances where geometric intuition can go wrong when applied to cardinalities of sets.