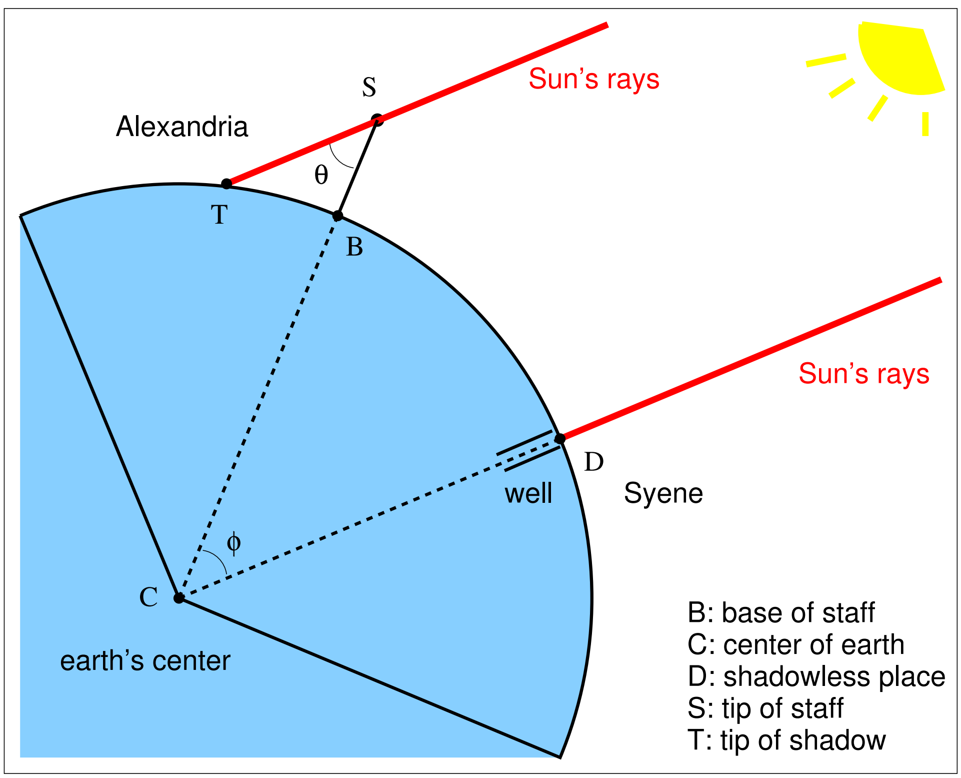
Measuring the circumference of the earth

It wouldn't be a history of math course without showing how the Greek mathematician Erastosthenes (276–194 BCE) calculated the size of the earth.

First of all, the ancient Greeks (and they weren't the only ones) knew perfectly well that the earth was round. Looking at ships' masts as they disappeared over the horizon was one piece of evidence. Lunar eclipses (where the earth is directly between the sun and moon) are further evidence: you can see the earth's curved shadow falling on the moon. Once you believe that the earth is curved, a sphere is the simplest possibility. They also knew that the sun was very very very very far away from the earth (our modern measurement is about 93 million miles).

So Erastosthenes started by assuming the earth was a sphere. He assumed that because the sun was so far away, the sun's rays striking two places on earth that weren't very far away from each other were nearly parallel. He also knew that the sun shone directly into a particular well at Syene, Egypt (in modern-day Libya) at the summer solstice. He knew that Alexandria was 5000 stadia (a unit of length; the singular is stadium) north of Syene, and that a staff in Alexandria cast a short shadow when the sun was at its zenith.[1] Erastosthenes put all this information together in the following diagram.



The diagram, of course, is not at all to scale—the staff is much smaller in real life than in the diagram. Be that as it may, Eratosthenes made the following observations about the diagram.

First, the fraction of the earth's circumference swept out by the arc BD is precisely $\phi/2\pi$ (if we measure ϕ in radians). We know that the arc measure is 5000 stadia, so this reduces the problem to figuring out the value of ϕ.

Second, while we obviously can't determine the angle ϕ directly (without a very powerful drill!), it must be almost equal to θ, because the two red lines (the sun's rays) are very close to parallel. (This is a direct application of Axiom 11 in the Math 409 list of axioms and theorems — if two parallel lines are cut by a third line, then alternate interior angles are equal. This fact is in Euclid's Elements and was certainly known to Eratosthenes.)

[1] That is, when the sun was as high as it would possibly get that day.

Third, we can determine θ from knowing SB and BT, which are easy to measure directly. In modern trigonometric notation,

$$\tan\theta = \frac{SB}{BT} \quad \text{or} \quad \theta = \arctan\left(\frac{SB}{BT}\right).$$

Eratosthenes would not have expressed this equation using the term "arctangent", but he was able to determine angles from their tangents, at least approximately. He found that θ was $1/50$ of a complete angle (i.e., $2\pi/50$ or $\pi/25$), which meant that the earth's circumference was $50 \cdot 5000 = 250,000$ stadia.

We don't know how long Eratosthenes' stadium was. If he was using the Egyptian stadium of 515.7 ft[2], his estimate would have been remarkably accurate: 24419 miles, compared to our modern value of about 24900 miles. His stadium may have been a bit longer or shorter than this, but whatever its exact value is, it doesn't diminish the insightfulness of his method.

By the way, the great Indian mathematician Aryabhata I (476–500 CE) gave an even more accurate estimate of the earth's circumference: 24,835 miles in our modern units. This remained the best measurement for over a thousand years.

For further reading:

- Biography of Eratosthenes at MacTutor
- Wikipedia article on the history of geodesy
- Biography of Aryabhata at MacTutor

[2] Close to 3 khet, if you're wondering.