The Four-Color Theorem

1 The seating problem

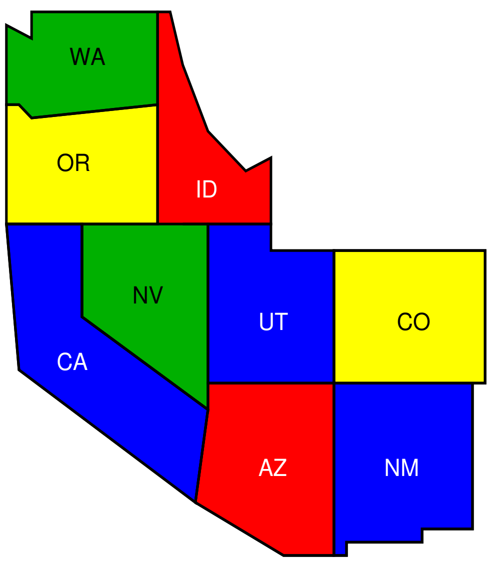
You are trying to figure out how to seat a group of guests at a party. The situation is complicated by a number of enmities between guests. You would like to avoid potential fistfights by seating known enemies at different tables. If you have a complete list of who cannot sit with whom, how do you figure out who to seat at which table?

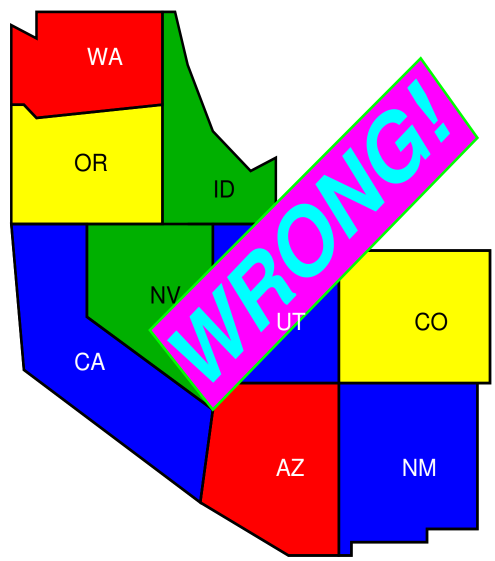
You could, for example, seat every single person at his or her own table, but that seems excessive — it would be better to have fewer tables and more people at each table. In fact, how can you accomplish your goal with as few tables as possible?

(Notes: The tables can be of any size and shape. Also, it doesn't matter who sits next to whom within each table — all that matters is the set of people at that table.)

2 The map-coloring problem

Given a map (of the states of the USA, or the counties of England, or the school districts of Kansas, or...), you want to color each of the regions so that adjacent regions never receive the same color. For example, the map on the left is colored correctly. The map on the right is not correct, because Nevada and Idaho, which share a little bit of border, are both colored green.





How many colors are necessary?

Of course, the answer to this question depends on the particular map. So what the problem is really asking is this: Is there some number of colors that is *always* enough, no matter what the map looks like?

We have to be a little more precise. First, by "adjacent" we mean "sharing a common piece of border", not just "meeting at a point". (For example, in the maps above, it is okay that Utah and New Mexico are both colored blue.) Otherwise, it would be easy to construct maps that required arbitrarily high numbers of colors (imagine a pizza sliced into a large number of wedges, all meeting at the center), so the problem wouldn't be very interesting.

Second, we have to assume that each region is contiguous — for example, the various islands of Hawaii are not considered part of the same region, and are allowed to have have different colors. Again, if we allow non-contiguous regions then the problem would not have a well-defined answer.

Even with these caveats, it's not too hard to construct maps for which at least four colors are required. The map above is an example. Here's one reason why. Each two of California, Nevada and Arizona share a common border, so they need three different colors — say blue, green and red respectively, as in the left-hand figure. If we don't want to use a fourth color, then Utah has to be blue (because it has a red neighbor and a green neighbor) and then Idaho has to be red (because it has a blue neighbor and a green neighbor). But then we're stuck when we try to color Oregon: it has a neighbor of each color, so we have no choice but to introduce a fourth color.

3 A reformulation using graph theory

You may have observed that the hosting problem and the map-coloring problem are similar. Guests correspond to regions of the map. Tables correspond to colors. Two people that are enemies are like two states that share a border. Assigning every guest a table so that enemies do not sit at the same table corresponds to assigning every region a color so that no two bordering states get the same color.

Both problems can expressed using graph theory. Remember that a graph consists of a set of vertices and a set of edges, where each edge joins two of the vertices. In the map-coloring problem, we can define a graph whose vertices are the regions of the map, and whose edges are the pairs of regions sharing a common border. For the map above, the graph has vertices

WA OR CA ID NV UT AZ CO NM

and edges

WA–OR	WA–ID	OR–ID	OR–CA	OR–NV
CA–NV	CA–NM	ID–NV	ID–UT	NV–UT
NV–AZ	UT–AZ	UT–CO	AZ–NM	CO–NM

In the seating problem, the vertices are the people at the party, and two people who are enemies are joined by an edge.

In the seating problem, the graph could be anything whatsoever. However, in the map-coloring problem, the graph has to be planar. That is, it can be represented by a diagram in the plane (dots for vertices, line segments or curves for edges) in which no two edges cross. (The border between, say, California and Nevada does not cross the border between Kansas and Missouri.) This is most emphatically not the case for all graphs.

Question for further thought: If you are given a list of vertices and edges of a graph, how can you tell if a graph is planar? For example, here is an old brainteaser. Three houses A, B, C need to be wired to three utilities X, Y, Z (say water, electricity, and phone lines). Can the wiring be done in such a way that no pair of wires cross? Stay tuned.

Meanwhile, what is a coloring? It's a function whose domain is the set of vertices and whose range is the set of colors. For example, the first coloring above is equivalent to the function k defined by

$$k(\text{WA}) = \text{green}, \quad k(\text{OR}) = \text{yellow}, \quad k(\text{CA}) = \text{blue}, \quad k(\text{NV}) = \text{green}, \quad \ldots$$

In the seating problem, we might just number the tables $1, 2, \ldots$ and let $k(P)$ be the table that person P is assigned to. So the coloring function k now has domain the set of guests and range the set of natural numbers.

A coloring function k is defined to be "proper" if $k(x) \neq k(y)$ whenever x and y form an edge. So the graph-theoretic restatement of the map-coloring problem is as follows:

Is there a number n such that every planar graph has a proper coloring with n or fewer colors?

4 The solution

It turns out that **four colors are enough to color any planar graph**. This fact, known as the **Four-Color Theorem (or 4CT)**, has a long and complicated (you might say colorful) history.

- 1852: English student Francis Guthrie proposes the problem. Prominent English mathematicians of the time, such as Augustus De Morgan and Arthur Cayley, get interested.
- 1879: Alfred Kempe publishes a proof of the Four-Color Theorem.
- 1890: Percy Heawood finds a subtle flaw in Kempe's proof. On the other hand, Heawood is able to prove that five colors are always enough.
- 1890–1976: Lots of people publish proofs, some serious, many cranky[1], all of them flawed in some way or other On the plus side, large areas of graph theory (perfect graphs, chromatic polynomials, ...) are developed in an attempt to prove the 4CT; these ideas turn out to be useful in totally different contexts.
- 1976: Kenneth Appel and Wolfgang Haken publish the first widely accepted proof of the 4CT. Their proof relies partly on some of Kempe's ideas, but is controversial because they reduce the theorem to checking "only" 1,936 cases, which requires a computer. Debate rages over whether Appel and Haken's argument constitutes a valid proof. As time goes on, the mathematical community comes to accept the method (even though mistakes are found in the details). Computer-aided proof becomes more and more a part of research mathematics.
- 1996: Robertson, Sanders, Seymour and Thomas publish an improved proof, with the same general structure as the Appel-Haken proof, but simpler and involving fewer cases. They also provide an efficient algorithm for coloring any planar graph. Source code for checking the cases is available on the Web, so you can "prove" the theorem yourself at home. For the juicy details, see http://www.math.gatech.edu/~thomas/FC/fourcolor.html.

[1]Probably because of its simplicity and notoriety, the 4CT is one of those theorems that everyone tries to prove, not unlike Fermat's Last Theorem. The recycling bins of mathematics departments and journals are full of rejected manuscripts that claim to give proofs of the 4CT.