Notes on polyhedra and 3-dimensional geometry

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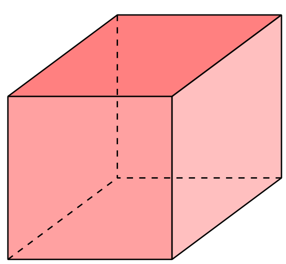
1 Polyhedra

Three-dimensional geometry is a very rich field; this is just a little taste of it. Our main protagonist will be a kind of solid object known as a **polyhedron** (plural: **polyhedra**). Its characteristics are:

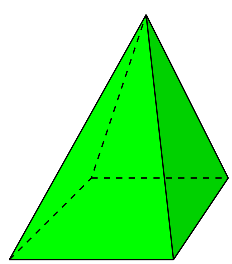
- it is made up of polygons glued together along their edges
- it separates $\mathbb{R}^3$ into itself, the space inside, and the space outside
- the polygons it is made of are called *faces*.
- the edges of the faces are called the edges of the polyhedron
- the vertices of the faces are called the vertices of the polyhedron.

The most familiar example of a polyhedron is a cube. Its faces are squares, and it has 6 of them. It also has 12 edges and 8 vertices.

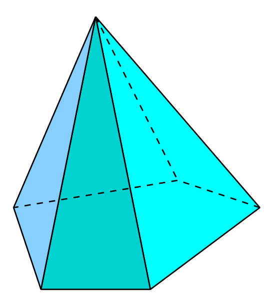
Another familiar example is a pyramid. A pyramid has a bottom face, which can be any polygon (you are probably most familiar with pyramids that have square bottoms), and the rest of its faces meet in one point.



Cube



Square pyramid

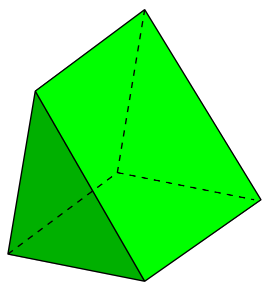


Pentagonal pyramid

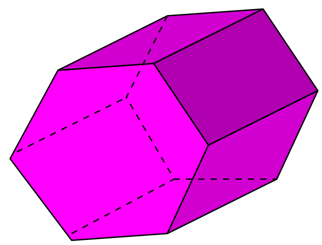
Problem #1 If the bottom face of a pyramid has n sides, how many faces, edges, and vertices will it have?

Another familiar example is a prism, which is a polyhedron with two congruent parallel faces in which the other faces are rectangles. The two congruent faces can be triangles, quadrilaterals, or anything else.

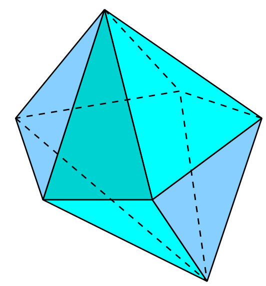
Another, perhaps slightly less familiar example is a bipyramid, which is built by taking two pyramids with congruent bases and gluing the bases together, so that only the triangular faces are left.



Triangular prism



Hexagonal prism



Pentagonal bipyramid

Problem #2 If one of the parallel faces of a prism has n sides, how many faces, edges, and vertices will the prism have?

Problem #3 If the base (or "equator") of a bipyramid has n sides, how many faces, edges, and vertices will the bipyramid have?

While there are lots of different polyhedra, they all have some common features just by virtue of being polyhedra.

Theorem 1. *In any polyhedron,...*

- *Every vertex must lie in at least three faces. (Otherwise, the polyhedron collapses to have no volume.)*
- *Every face has at least three vertices. (It's a polygon, so it better have at least three sides.)*
- *Every edge must lie in exactly two faces. (Otherwise, the polyhedron wouldn't have an inside and an outside.)*

As usual, you can learn a lot by playing with non-standard examples. For example:

Problem #4 Can you construct a polyhedron with two parallel faces, one a triangle, the other a rectangle?

Problem #5 Can you construct a polyhedron so that exactly one face is not a quadrilateral?

Problem #6 Can you construct a polyhedron so that exactly one face is not a rectangle?

Problem #7 Can you construct a polyhedron in which every face is a hexagon?

2 Euler's formula

Let v, e, and f be the numbers of vertices, edges and faces of a polyhedron. For example, if the polyhedron is a cube then $v = 8$, $e = 12$ and $f = 6$.

Problem #8 Make a table of the values for the polyhedra shown above, as well as the ones you have built. What do you notice?

You should observe that $v - e + f = 2$ for all these polyhedra. This relationship is called *Euler's formula*, and it is vitally important in geometry, topology, and many other areas of mathematics. (The idea of adding up things in different dimensions, counting even dimensions as positive and odd dimensions as negative, just comes up everywhere.)

Here is a cute proof of Euler's formula, from p. 198 of George E. Martin (no relation), *Transformation Geometry: An Introduction to Symmetry.* (Springer-Verlag, 1982, New York).

> To prove the famous formula, imagine that all the edges of a convex polyhedron are dikes, exactly one face contains the raging sea, and all other faces are dry. We break dikes one at a time until all the faces are inundated, following the rule that a dike is broken only if this results in the flooding of a face. Now, after this rampage, we have flooded $f - 1$ faces and so destroyed exactly $f - 1$ dikes. Noticing that we can walk with dry feet along the remaining dikes from any vertex p to any other vertex along exactly one path, we conclude there is a one-to-one correspondence between the remaining dikes and the vertices excluding p. Hence there remain exactly $v - 1$ unbroken dikes. So $e = (f - 1) + (v - 1)$ and we have proved Euler's formula.

This is a vivid, dramatic proof. But is it correct?

If we take away the colorful imagery, the proof boils down to four assertions about what happens after all the dikes are broken.

Assertion #1: We have flooded $f - 1$ faces.

Assertion #2: We can walk with dry feet along the unbroken edges from any vertex to any other vertex.

Assertion #3: There is a unique path consisting of unbroken edges connecting any two vertices.

Assertion #4: There are exactly $v - 1$ unbroken edges.

Our task is to show that (i) these assertions are correct and (ii) they imply Euler's formula.

Proof of assertion #1: The only face we have not flooded is the one which originally contained the raging sea.

Proof of assertion #2: What if there is some pair of vertices that are cut off from each other? Then there must have been some bridge B such that the dike network was connected just before B was broken, and disconnected just after B was broken. But that could only happen if the raging sea had already flooded *both* sides of B — and in that case we wouldn't have broken B in the first place. So it's impossible for there to be two mutually un-walkable-between vertices.

Proof of assertion #3: Suppose, by way of contradiction, that there were two different paths between some pair of vertices. But then those paths would enclose some unflooded region, and that means that we didn't break enough bridges.

Proof of assertion #4: Pick a vertex p. There are exactly $v - 1$ vertices other than p, and for each other vertex q, there is a unique edge $f(q)$ that is the first step from q to p. If q and q' are different vertices, then it cannot be the case that $f(q) = f(q')$ (otherwise one or both of the paths to p involves doubling back). On the other hand, every edge is $f(q)$ for *some* q (because the network is connected, so it's possible to walk from p towards that edge and eventually cross it). That is, the function

$$f: \{\text{vertices other than } p\} \to \{\text{edges}\}$$

is a bijection, and so $e = v - 1$.

2.1 Inequalities from Euler's formula

When combined with other observations about the numberse v, e and f, Euler's formula has other consequences. Remember that the *degree* of a vertex is the number of edges attached to it. For instance, all vertices in a cube have degree 3, while all vertices in an octahedron have degree 4. The degrees don't have to be the same in an arbitrary polhedron: In a pentagonal pyramid (see p. 1), the apex of the pyramid has degree 5, while each of the base vertices has degree 3.

If you add up all the degrees of vertices in a polyhedron (in fact, in any graph), each edge will be counted twice. That is,

$$(\text{degree of vertex }\#1) + (\text{degree of vertex }\#2) + \cdots + (\text{degree of vertex }\#v) = 2e. \tag{1}$$

If you add up the numbers of edges in all the faces of a polyhedra, you will again count each edge twice (because each edge lies in exactly two adjacent faces). That is,

$$(\text{number of edges in face }\#1) + (\text{number of edges in face }\#2) + \cdots + (\text{number of edges in face }\#\text{f}) = 2e. \tag{2}$$

For example, a cube has 8 vertices of degree 3 each (so the sum of all degrees is 24), and 6 quadrilateral faces (so the sum in (2) is also 24), and 12 (= 24/2) edges.

Formula (1) goes by the name of the **Handshaking Theorem** in graph theory — if you think of the vertices as people and each edge as a handshake between two people, then the degree of v is the number of people with whom v shakes hands, so the theorem says that adding up those numbers for all people, then dividing by 2, gives the total number of handshakes.

Problem #9 Verify that these general rules are true for your favorite polyhedra.

Now every face has to have at least 3 sides. So the sum in (2) has to be *at least* $3f$. this observation together with (2), it tells us that

$$3f \leq 2e \qquad \text{or equivalently} \qquad f \leq \frac{2e}{3}.$$

Now, substitute this inequality into Euler's formula to get rid of the f:

$$\begin{aligned} 2 = v - e + f &\leq v - e + \frac{2e}{3} \\ &= v - \frac{e}{3} \end{aligned}$$

or

$$6 \leq 3v - e$$

or

$$e \leq 3v - 6.$$

Also, every vertex has to have degree at least 3, so the same calculation says that

$$e \leq 3f - 6$$

or equivalently

$$\frac{2e}{f} \leq 6 - \frac{12}{f} < 6.$$

So what? Well, $2e/f$ is the average number of edges in a face (just because there are f faces in total, and the sum of their numbers of edges is $2e$). Therefore, we have proved:

Theorem 2. *In every polyhedron, the average number of sides in a face is less than 6.*

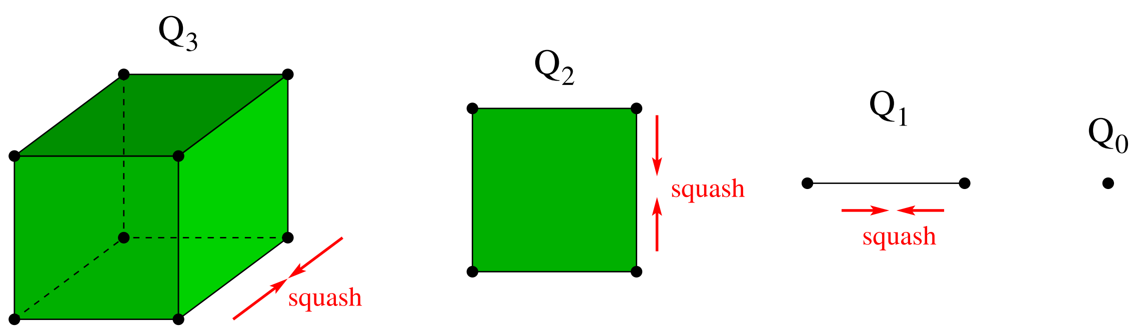
In particular, it's impossible to build a polyhedron all of whose faces are hexagons!

3 Cubes, cubes and more cubes

What does a 4-dimensional cube look like?

This is a scary-sounding question — how can you see anything in 4-dimensional space? But if we work by analogy and use what we can see about ordinary (i.e., visible) 1-, 2- and 3-dimensional spaces, then we can get a pretty good idea.

We know what a 3-dimensional cube looks like. What's a 2-dimensional cube? It's a square — the thing you get by pressing a cube flat. And if you press the squuare flat, you get a line segment (a 1-dimensional cube), and if you press the line segment flat, you get a single point (an 0-dimensional cube).

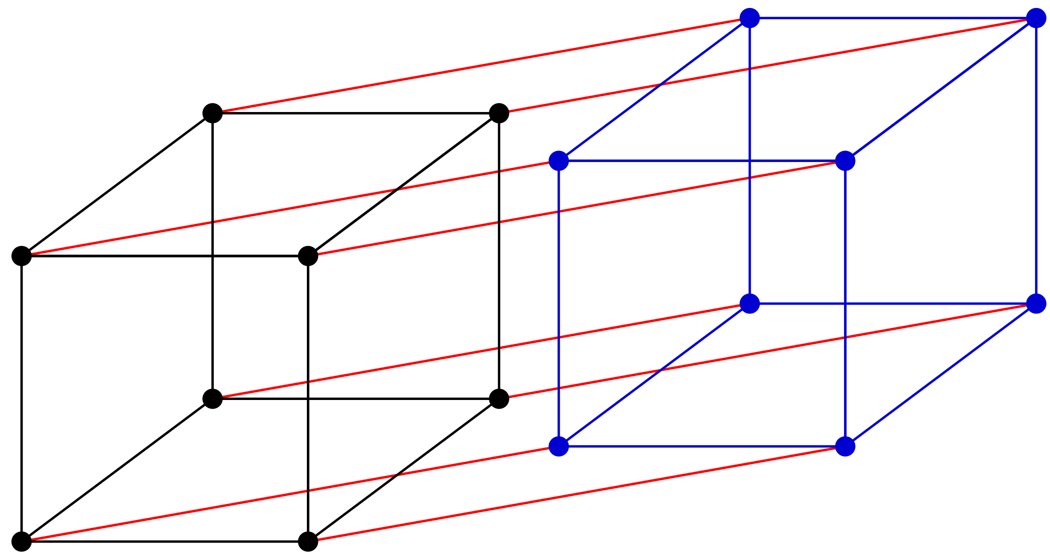


Of course, in the preceding paragraph, the word "cube" is being used more generally than it usually is; it's shorthand for "thing that is analogous to a cube but happens to live in a different-dimensional space." A common notation for the n-dimensional cube is Q_n: so Q_0 is a point, Q_1 is a line segment, Q_2 is a square, and Q_3 is the familiar three-dimensional cube.

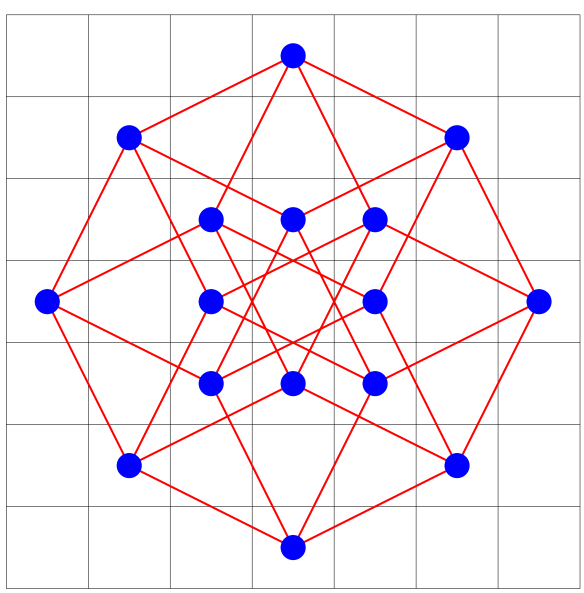
We've seen how to make smaller-dimensional cubes from bigger ones. What about the reverse process?

To make a square (Q_2) out of a line segment (Q_1), you can make two copies of the line segment and attach each corresponding pair of vertices with a new line segment. This same procedure lets you build a cube (Q_3) starting with a square (Q_2), or even a line segment (Q_1) from a point (Q_0).

So, what about Q_4? By analogy, you can build Q_4 by starting with two copies of Q_3 and attaching each corresponding pair of vertices with a line segment.



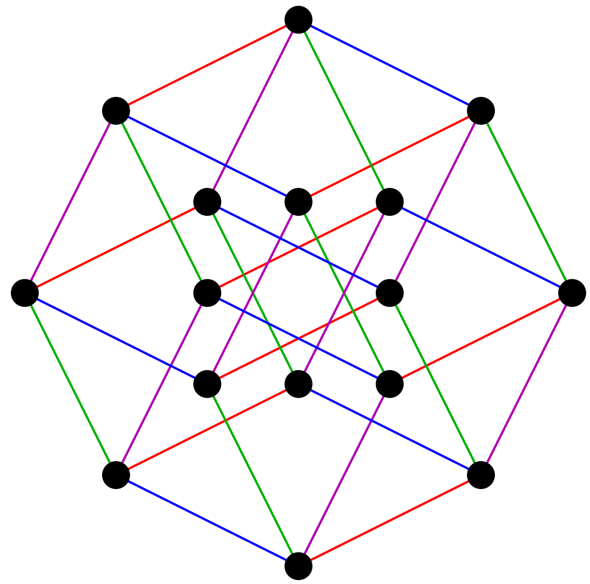
It's not so easy to see the symmetry in this picture, but miraculously there is a beautiful and surprising other way to draw Q_4. Make a 7×7 chessboard and place dots in the middle squares of the bottom and top rows. Then, draw in all the possible ways that a knight can move from one dot to the other in four moves. The result is Q_4!



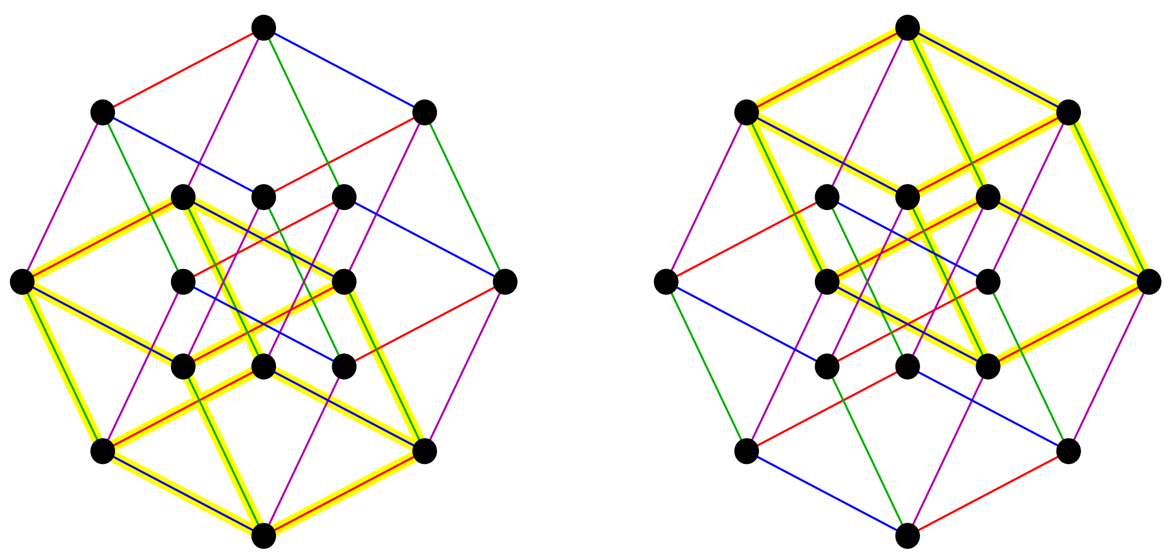
How many copies of Q_3 are there inside Q_4? Since

- there are two Q_0's inside Q_1 (i.e., a line segment has two points);
- there are four Q_1's inside Q_2 (i.e., a square has four sides);
- there are six Q_2's inside Q_3 (i.e., a cube has six faces);

if this pattern continues, then the answer should be eight. You can see this by coloring the edges of Q_4, like this:



Now each set of three colors (and there are four such sets) can be used to make two Q_3's. For example, here are the two Q_3's with red, green and blue (but no purple) edges:



In general, how many copies of Q_k sit inside Q_n? If we call this number $f(n,k)$ and make a table of values, we notice a variety of wonderful patterns:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	1	0	0	0	0	0
$n=1$	2	1	0	0	0	0
$n=2$	4	4	1	0	0	0
$n=3$	8	12	6	1	0	0
$n=4$	16	32	24	8	1	0
$n=5$	32	80	80	?	10	1

Here are some of the patterns:

- $f(n,n)=1$. This is pretty simple; everything contains one copy of itself!

- $f(n,0)=2^n$. A point has 1 vertex, a line segment has two, a square has four, a 3D cube has eight, ... In general, since you can build Q_n out of two copies of Q_{n-1}, it makes sense that the number of vertices doubles each time.

- $f(n,n-1)=2n$. We already noticed this.

- What about $f(n,1)$, i.e., the number of edges (i.e., line segments)? This is the $k=1$ column of the table. The pattern is not as obvious, but it turns out to be $f(n,1)=n\cdot 2^{n-1}$. Here's why. Each vertex in Q_n has degree n, and there are 2^n vertices, so adding up all the degrees gives $n\cdot 2^n$. This, we know, is twice the number of edges. Therefore, the number of edges re are $(n\cdot 2^n)/2 = n\cdot 2^{n-1}$ edges.

- What about $f(n,2)$, i.e., the number of squares in Q_n? This is the $k=2$ column of the table, and the pattern is even less obvious, but fortunately we can use a similar counting technique. First of all, every vertex belongs to n edges, and each of those pairs of edges forms a square. So every vertex belongs to $\binom{n}{2}=n(n-1)/2$ squares, and multiplying this by the number of vertices gives $(n(n-1)/2)\cdot 2^n = n(n-1)\cdot 2^{n-1}$. On the other hand, we have now counted each square four times — once for each of its corner vertices. Therefore, the actual number of squares in Q_n is

$$f(n,2)=\frac{n(n-1)\cdot 2^{n-1}}{4}=n(n-1)\cdot 2^{n-3}.$$

Using these observations, we can fill out almost all of that table, except for the question mark for $f(5,3)$.

What about Euler's formula? In this notation, the relation $v-e+f=2$ for the cube Q_3 becomes

$$f(3,0)\ -\ f(3,1)\ +\ f(3,2)\ =\ 2.$$

This is true: $8-12+6=2$. What if we look at the alternating sum of numbers for each row (ignoring the 1's)?

$$\begin{array}{llllllllllll}
n=1: & f(1,0) & & & & & & & & = & 2 & = & 2 \\
n=2: & f(2,0) & - & f(2,1) & & & & & & = & 4-4 & = & 0 \\
n=3: & f(3,0) & - & f(3,1) & + & f(5,2) & & & & = & 8-12+6 & = & 2 \\
n=4: & f(4,0) & - & f(4,1) & + & f(5,2) & - & f(5,3) & & = & 16-32+24-8 & = & 0 \\
n=5: & f(5,0) & - & f(5,1) & + & f(5,2) & - & f(5,3) & + f(5,4) & = & 32-80+80-?+10 & = & \ldots
\end{array}$$

If this pattern continues (and it does!), then the alternating sum for $n=5$ should be 2, which means that the question mark should be $f(5,3)=40$.

In fact, this is a general rule about polyhedra in all dimensions! For any n-dimensional polyhedron, the alternating sum

$$\begin{array}{rl} & \text{number of vertices (0-dimensional pieces)} \\ - & \text{number of edges (1-dimensional pieces)} \\ + & \text{number of faces (2-dimensional pieces)} \\ - & \text{number of 3-dimensional pieces (whatever they're called)} \\ & \vdots \\ \pm & \text{number of } (n-1)\text{-dimensional pieces} \end{array}$$

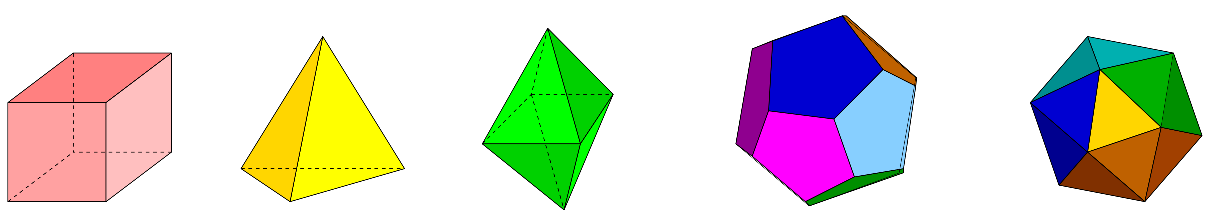
comes out to either 2 (when n is odd) or 0 when n is even).

4 The Platonic solids

Definition 1. A polyhedron is called *regular* (or a *Platonic solid*) iff (a) all of its faces are congruent; (b) all of its faces are regular polygons; and (c) each of its vertices meets the same number of edges as every other vertex.

So cubes are Platonic, but most prisms and pyramids are not. You may have heard of the following Platonic solids besides the cube:

- tetrahedron (the faces are 4 equilateral triangles);
- octahedron (8 equilateral triangles);
- dodecahedron (12 regular pentagons);
- icosahedron (20 equilateral triangles).



Cube Tetrahedron Octahedron Dodecahedron Icosahedron

You haven't heard of any others because...

Theorem 3. *There are exactly five Platonic solids.*

This is a surprising theorem. Frequently when we define a nice class of objects in mathematics it is large, in fact usually infinite (think of the set of primes, the set of isometries of the plane, the collection of all regular polygons...). But this class is not only finite; it has only five things in it.

One part of the proof — that there are at least five Platonic solids — is already done. We know what they are. In fact you've built them out of Zometool. So the interesting part is proving that there aren't any more. In order to do this, you need to think carefully about how you fit polygons together to make polyhedra — both the numerical observations that we made above (see Theorem 1), and some geometric facts: for example, thinking about the angles that faces meet at.

Problem #10 Fold a piece of paper. Cut it so that you have two congruent polygons joined at the fold. Let p be a point at the edge of the fold, and let ℓ, m be the sides through p which do not coincide with the fold. Fold and unfold the paper so that you narrow and expand the angle between ℓ and m at p. When is this angle the biggest possible? If the two faces were a piece of a polyhedron, would you need at least one more face at p? Could you have more than one more face at p?

Problem #11 Fit some polygons together at a vertex p to start a polyhedron What is the sum of the angles touching p? Now flatten the part touching p (you will have to cut at least one edge to do this). No matter what your polyhedron is, can the sum of the angles touching p sum up to more than $360°$? Can they sum up to exactly $360°$? Explain briefly.

Platonic solids are very special, because each vertex must belong to the same number of faces, say n, and each face must be a polygon with the same number of sides, say s. What can we say about these numbers?

We already know that $s \geq 3$ and $n \geq 3$ (see Theorem 1).

Each face F of $\mathcal{P}$ is a regular polygon with s sides and s angles. The sum of the angles of F is $180(s-2)$, so each single angle measures $180(s-2)/s$.

Therefore, if we consider the n faces that fit together at a single vertex, we see that their angles add up to

$$\frac{180(s-2)n}{s}.$$

On the other hand, we know that this quantity has to be $< 360°$. If n and s are too large, then this condition will fail. So we can figure out all the possibilities just by brute force:

	$n=3$	$n=4$	$n=5$	$n=6$	$\cdots$
$s=3$	180	240	300	360	$\cdots$
$s=4$	270	360	450	$\cdots$	
$s=5$	324	432	$\cdots$		
$s=6$	360	$\cdots$			
$\vdots$	$\vdots$				

There are evidently only five possibilities:

- the faces are equilateral triangles ($s=3$); the number n of faces meeting at a vertex may be 3 (the tetrahedron), 4 (the octahedron), or 5 (the icosahedron).
- the faces are squares ($s=4$); the number of faces meeting at a vertex must be 3 (the cube).
- the faces are regular pentagons ($s=5$); the number of faces meeting at a vertex must be 3 (the dodecahedron).

This almost proves the theorem. Except, how do you know that the parameters s and n determine the polytope?

Problem #12 Suppose you have a bunch of equilateral triangles, squares, or regular pentagons. Suppose you are told how many of such faces each vertex of a regular polyhedron must meet. Suppose your constraints are consistent with conclusion 2. Then there is exactly one way to construct the regular polyhedron, and it must be one of those listed in conclusion 2.