Math 409, Spring 2013
Homework/In-Class Problems (updated 5/1/13)

1 Ruler-and-compass constructions

Unless otherwise specified, do each construction on its own page. (It is okay to use the front and back of the same piece of paper for different constructions.) Always make your constructions big enough that all the parts of the drawing are easily readable.

RC 1. Draw a straight line segment $\overline{AB}$. Draw a point P somewhere else on the page. Now construct a line segment of exactly the same length as $\overline{AB}$ with P as one endpoint. Label the new line segment PQ.

RC 2. Draw a triangle ΔABC. Now copy it exactly somewhere else on the page. Label the new copy ΔPQR.

RC 3. Construct an equilateral triangle. Label its vertices A, B, C.

RC 4. Draw an angle $\angle ABC$. Draw a point P somewhere else on the page. Now construct an angle of exactly the same measure as $\angle ABC$, with P as its vertex.

RC 5. Draw an angle $\angle ABC$. Now bisect it.

RC 6. Draw a straight line segment $\overline{AB}$. Now construct a line perpendicular to $\overline{AB}$ that bisects its length. Label the line L.

RC 7. Draw a line L and a point P that does not lie on L. Now construct a line through P that is perpendicular to L. Label this line M. (Hint: Use RC 6.)

RC 8. Draw a line L and a point P that does not lie on L. Now construct a line through P that is parallel to L. Label this line N. (Hint: Use RC 7. If you want, you can use the same sketch for RC 7 and RC 8, provided that both M and N are clearly labelled.)

RC 9. Construct a square. Label its vertices A, B, C, D.

RC 10. Construct a regular hexagon. Label its vertices A, B, C, D, E, F.

RC 11. Draw an angle. Bisect it.

RC 12. Draw a line segment $\overline{AB}$. Trisect it. That is, find a point C on $\overline{AB}$ such that $AC = \frac{1}{3}AB$. Can you extend your method to n-section?

2 Euclidean geometry

When you write proofs, you should refer to the master list of axioms, definitions and theorems on the course website, and cite (by number) every fact or definition that you use.

EG 1. Let ΔABC be an equilateral triangle. Let X, Y, and Z be the midpoints of the segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ respectively. Prove that ΔXYZ is equilateral. **Bonus:** State and prove a conjecture about what happens with if ΔABC is only assumed to be isosceles, rather than equilateral.

EG 2. (a) Explain why your construction of a perpendicular bisector in RC 6 works.

(b) Explain why your construction of an angle bisector in RC 11 works.

EG 3. Prove that the angles of a triangle sum to $180°$.

EG 4. Prove your observation about the angles in SA 4.

EG 5. (Thales' Theorem) Prove that the base angles of an isosceles triangle are congruent. That is, if ΔABC is a triangle and $AB = AC$, then prove that $m\angle ABC = m\angle ACB$.

EG 6. Let ΔABC be a triangle such that $m\angle BAC = m\angle ABC$. Prove that $AC = BC$. (Note: This is *not* the same thing as Thales' Theorem!)

EG 7. In SA 7, you started with an arbitrary triangle ΔABC and constructed its circumscribed circle. Explain why your construction works.

EG 8. Suppose that R is a rhombus inscribed in a circle (that is, it is possible to draw a circle containing all four points of the rhombus). Prove that R is in fact not just a rhombus, but a square. (Recall that a rhombus is a quadrilateral whose four sides all have equal length.)

EG 9. Prove the Vertical Angle Theorem: when two lines meet and form four angles, each pair of angles not sharing a common side has equal measure.

EG 10. Write a precise definition for the term "ray". Your definition may (in fact, should) use other axioms and definitions as needed.

EG 11. Prove that the interior angles of a quadrilateral add up to 360°.

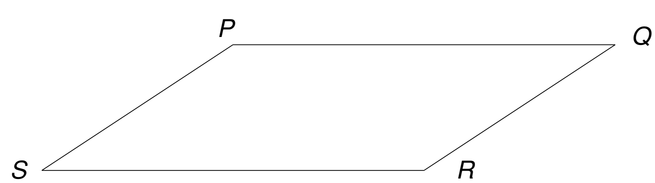
EG 12. Define the *distance of a point p from a line ℓ* to be the length of the line segment from p to a point on ℓ. Prove that a point is equidistant from the sides of an angle iff it is on the angle bisector.

Note: Remember that "iff" means "if and only if"—so you need to prove two separate things here: (1) *if* P is a point equidistant from the sides of an angle, *then* it is on the angle bisector; and
(2) *if* P is a point on the bisector of an angle, *then* it is equidistant from the sides of the angle.

EG 13. Use EG 12 to show that the three angle bisectors of a triangle meet in one point.

EG 14. Prove that the trisection of a line segment (the first part of SA 18) works; i.e., that $AT = \frac{1}{3}AE$. In your proof, use the names for points and lines that are given in SA 18. (Hint: Use similar triangles. You may need to draw one or more new lines, or to extend the lines that are already in the figure.)

EG 15. Let $PQRS$ be a parallelogram, that is, a quadrilateral such that $\overleftrightarrow{PQ}$ is parallel to $\overleftrightarrow{RS}$, and $\overleftrightarrow{PS}$ is parallel to $\overleftrightarrow{QR}$, as in the following figure.



a. Prove that opposite sides of the parallelogram have equal length, i.e., that $PQ = RS$ and $PS = QR$. (Hint: Use ASA.)

b. Prove that opposite angles are equal, i.e., that $m\angle PQR = m\angle RSP$ and $m\angle QRS = m\angle SPQ$. (Hint: Use part (a).)

EG 16. Prove that the diagonals of a parallelogram meet at a right angle iff the parallelogram is a rhombus.

(Remember that "iff" proofs require two separate arguments; see EG 12.)

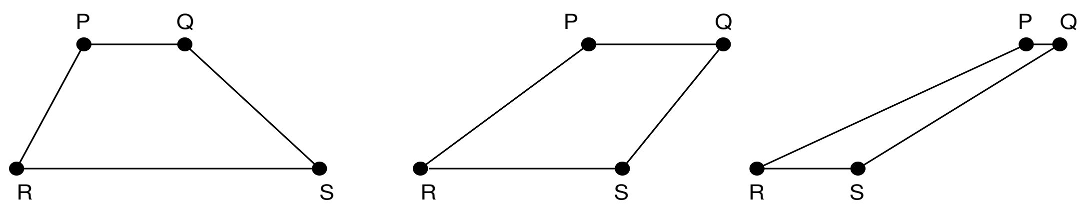
EG 17. Prove that the diagonals of a parallelogram are congruent to each other iff the parallelogram is a rectangle.

EG 18. Three circles in the plane, all of the same radius r, pass through a common point O. Show that their other points of intersection X, Y, Z lie on a circle of radius r.

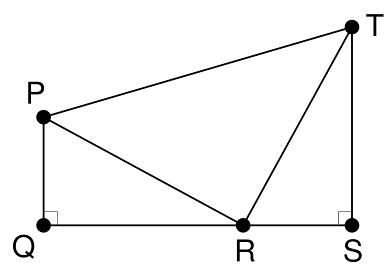
EG 19. A *trapezoid* is a quadrilateral $PQRS$ such that sides PQ and RS are parallel, but with no other restrictions.[1] Using only the axioms and theorems for area, prove the well-known formula for the area of a trapezoid:

$$\text{area}(PQRS) = \frac{h \cdot (PQ + RS)}{2}$$

where h is the distance between $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ (that is, the length of a segment with endpoints on, and perpendicular to both of, those lines). Note: Your proof should apply to all the trapezoids shown below — you may need multiple cases.



EG 20. The following proof of the Pythagorean theorem was discovered in 1876 by future U.S. President James A. Garfield. Construct a right triangle ΔPQR with sides PQ, QR of lengths a and b and hypotenuse PR of length c. Construct another right triangle ΔRST such that $\Delta PQR \cong \Delta RST$ and R is between Q and S, as shown below. With this setup, prove that $c^2 = a^2 + b^2$. (Hint: Use EG19.)



EG 21. Suppose that P is a parallelogram inscribed in a circle (that is, it is possible to draw a circle containing all four points of the parallelogram). Prove that P is in fact a rectangle.

(Compare this problem to EG 8.)

EG 22. Prove that every rhombus is a parallelogram! (This is something we have been assuming for a while, but never actually proved.)

EG 23. Let $ABCD$ be a quadrilateral. Let P, Q, R, S be the midpoints of line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively. Prove that quadrilateral $PQRS$ is in fact a parallelogram.

3 *Sketchpad* assignments

Do each problem in a fresh sketch and send it to me (jmartin@math.ku.edu) as an e-mail attachment. Be sure to name each sketch exactly as specified, and be sure to include your last name and the problem numbers

[1] So parallelograms, rectangles, rhombuses and squares are all special cases of trapezoids.

in the subject line of your e-mail. For example, if your name is Emmy Noether and you are handing in problems 11 and 15, your subject line should read “Noether SA 11, 15”.

In these problems, “draw” means to make an arbitrary figure, while “construct” means to build a figure based on previous objects. For example, if the instructions say “draw a triangle,” you should just put three independent points in your sketch and connect them with three line segments — you should not specify any relationship between the points.

SA 1. Using only the Point, Circle and Line tools from the toolbar, construct an equilateral triangle with vertices labeled A, B, C. Send it to me in a sketch named EQUILATERAL.

SA 2. Using only the Point, Circle and Line tools from the toolbar, construct a regular hexagon with vertices labeled A, B, C, D, E, F. Send it to me in a sketch named HEX.

SA 3. Using only the Point, Circle and Line tools from the toolbar, construct a square with vertices labeled A, B, C, D. Send it to me in a sketch named SQUARE.

SA 4. Draw a line segment AB. Construct its midpoint and label it C. Construct a circle centered at C that has AB as a diameter. Draw a point D on the circle. Measure the angles $\angle ADB$, $\angle CAD$ and $\angle BCD$. Now move the point D around. Send it to me in a sketch named SA4. Tell me what you observe about $m\angle ADB$. Tell me what you observe about $m\angle CAD$ and $m\angle BCD$ relative to each other.

SA 5. Start with the construction of SA 4. Now draw a point E anywhere on your worksheet, and measure $\angle AEB$. Move the point E around inside the circle and tell me what you observe about $m\angle AEB$. Now move E around outside the circle and tell me what you observe about $m\angle AEB$. Make a conjecture about how $m\angle AEB$ depends on the position of E.

SA 6. Start by drawing a triangle ΔABC.

a. In blue, construct the perpendicular bisectors of each side of the triangle. Move the points A, B, C around. Tell me on your sketch whether the blue lines meet in one, two or three points.

b. In green, construct the lines bisecting each angle of the triangle. Move the points A, B, C around. Tell me on your sketch whether the green lines meet in one, two or three points.

c. In red, draw lines connecting each vertex of the triangle to the midpoint of the opposite side. Move the points A, B, C around. Tell me on your sketch whether the red lines meet in one, two or three points.

Send your sketch to me in a file named TRIANGLE.

SA 7. Draw a triangle ΔABC. Construct its circumscribed circle O that is, a circle which passes through all three points A, B, C. (Do not use any of the circle or arc constructions from the Construct menu.) Move the triangle around to make sure that your construction works. Send your sketch to me in a sketch named CIRCUMCIRCLE.

SA 8. Start with the construction of SA 7. Label the center of the circle Z. Measure the angle $\angle ABC$ and move the triangle around some more, so that Z will move around too. Tell me what the measure of the angle is when Z is (a) inside ΔABC; (b) on segment $\overline{AC}$; (c) outside ΔABC.

SA 9. Draw a rectangle Q. Construct a new quadrilateral R whose vertices are the midpoints of the sides of Q. Tell me what kind of quadrilateral R is (as specifically as possible). Give a proof that your answer is correct.

SA 10. Draw a rhombus Q. Construct a new quadrilateral R whose vertices are the midpoints of the sides of Q. Tell me what kind of quadrilateral R is (as specifically as possible). Give a proof that your answer is correct.

SA 11. Draw an arbitrary quadrilateral Q. Construct a new quadrilateral R whose vertices are the midpoints of the sides of Q. Tell me what kind of quadrilateral R is (as specifically as possible). Give a proof that your answer is correct.

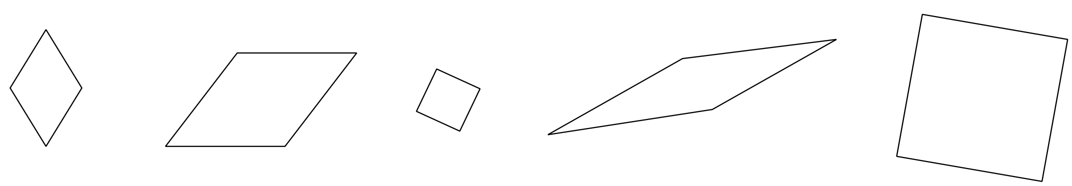
SA 12. Start by drawing a circle O. Put four points A, B, C, D on O and connect them to form a quadrilateral. Use *Sketchpad* to calculate $m\angle ABC + m\angle ADC$. What happens when you move the points around?

SA 13. Draw a circle and put a point on it. Construct a line which is tangent to the circle at the point you drew.

SA 14. Draw a triangle ΔABC and use Sketchpad to measure the side lengths. Construct the angle bisector of $\angle BAC$, and color it green. Find the midpoint X of $\overline{BC}$, construct the segment $\overline{AX}$, and color it red. Move the points around. Make a conjecture about what happens when the red and green lines (or line segments) overlap. The conjecture should be stated precisely

SA 15. Draw a triangle ΔABC. Now construct its inscribed circle, that is, a circle which touches every side of the triangle but does not cross any side. Move the triangle around to make sure that your construction works. Send your sketch to me in a sketch named INCIRCLE.

SA 16. Construct a rhombus $ABCD$. (A rhombus is a quadrilateral with four sides of equal length.) To earn full credit, you should be able to drag the vertices around so that the lengths and angles can vary freely. For example, you should be able to make your figure look like any possible rhombus (such as all of the following):



Send your construction to me in a sketch named RHOMBUS. Before you send it, hide everything except the four points and four sides of the rhombus.

Note: One idea you may have is to draw a rectangle and connect the midpoints. However, I don't think you'll be able to get all possible rhombi this way (although you should feel free to try and prove me wrong).

SA 17. Draw a circle, label its center O, and draw a chord of the circle (that is, a line segment whose endpoints lie on the circle). Call those endpoints A and B. Now place two points C and D on the circle, one on each side of the chord. Measure the angles $\angle ACB$, $\angle ADB$, and $\angle AOB$. Then answer the following questions.

a. Move A around the circle while keeping the other points fixed, and while keeping C and D on different sides of the chord $\overline{AB}$. Tell me how the angles change as you do so.

b. Move C and D around, while keeping them on different sides of the chord $\overline{AB}$. Tell me how the angles change as you do so.

c. How do you know which of $\angle ACB$ and $\angle ADB$ is bigger? When are they equal?

d. Tell me an equation that you think relates $\angle ACB$ and $\angle ADB$.

e. Tell me an equation that you think relates $\angle AOB$ to the *smaller* of $\angle ACB$ and $\angle ADB$.

Write your answers in a text box in your sketch. Send the sketch to me under the name CIRCLEANGLES.

SA 18. *Trisection of a line segment.* Draw a line segment $\overline{AE}$. Construct a new line ℓ somewhere else in your sketch, and construct points W, X, Y, Z appearing in that order on ℓ, so that $WX = XY = YZ$. Then find a point D so that (i) $AE = AD$ and (ii) $\overleftrightarrow{AW}$ is parallel to $\overleftrightarrow{DZ}$.

To find the points W, X, Y, Z, D, you'll probably have to draw a bunch of circles, which you should hide after you finish constructing the points.

Now, construct a line m parallel to $\overleftrightarrow{AW}$ containing X, and let B be the point where m meets $\overleftrightarrow{AD}$. Then construct a line n parallel to $\overleftrightarrow{AW}$ containing Y, and let C be the point where n meets $\overleftrightarrow{AD}$.

Next, use Sketchpad to convince yourself that $AB = BC = CD = \frac{1}{3}AD$. (Move the points around, and do some measurement.) Now you should be able to find points T, U on $\overline{AE}$ such that $AT = TU = UE = \frac{1}{3}AE$. Do *not* send me this sketch.

Does this work for all original segments A, E? Not quite — if you drag the point E around, you can make the points B, C, D disappear. So the first problem is to think about how to choose W, X, Y, Z in terms of A and E so that this problem doesn't occur.

Finally, construct a new sketch with a "quintasection", in which you draw an arbitrary line segment AF and divide it into five equal-length segments AB, BC, CD, DE, EF, adapting the method you've just learned. Send me this sketch under the name FIVE. (This sketch is the only thing you have to hand in for this problem.)

SA 19. Draw an arbitrary triangle $\mathcal{T}$. Construct the midpoints of the sides of $\mathcal{T}$ and connect them to form a new triangle $\mathcal{M}(\mathcal{T})$. Construct the medians[2] of $\mathcal{T}$ and the midpoints of the sides of $\mathcal{M}(\mathcal{T})$.

a. Send me this sketch under the name MID.

b. Tell me the relationship between the midpoints of $\mathcal{M}(\mathcal{T})$ and the medians of $\mathcal{T}$.

c. Tell me the relationship between the medians of $\mathcal{M}(\mathcal{T})$ and the medians of $\mathcal{T}$.

d. Tell me an equation that relates the area of $\mathcal{M}(\mathcal{T})$ and the area of $\mathcal{T}$. (To measure the area of a triangle in Sketchpad, use the arrow tool to select its points. Then select **Interior** from the **Construct** menu (which will make the triangle change color) and select **Area** from the **Measure** menu.)

Repeat this construction five more times to make triangles $\mathcal{T}_0 = \mathcal{T}$; $\mathcal{T}_1 = \mathcal{M}(\mathcal{T}_0)$; $\mathcal{T}_2 = \mathcal{M}(\mathcal{T}_1)$; $\mathcal{T}_3 = \mathcal{M}(\mathcal{T}_2)$; $\mathcal{T}_4 = \mathcal{M}(\mathcal{T}_3)$; $\mathcal{T}_5 = \mathcal{M}(\mathcal{T}_4)$. (Don't send me a sketch of this.)

e. Tell me the relationship between the medians of all these triangles.

f. Tell me an equation that relates the area of $\mathcal{M}(\mathcal{T}_n)$ and the area of $\mathcal{T}$.

SA 20. Use Sketchpad to convince yourself that there is no "SSA congruence theorem". That is, construct two triangles ΔABC, $\Delta A'B'C'$ such that

$$AB = A'B', \quad BC = B'C', \quad \text{and} \quad m\angle BAC = m\angle B'A'C', \qquad \text{but} \qquad \mathbf{\Delta ABC \ncong \Delta A'B'C'}.$$

(Hints: It is okay if $A = A'$ and $B = B'$; in fact, it will make the construction easier to understand. So you need to construct a segment AB and two points C, C' such that $BC = BC'$ and $m\angle BAC = m\angle BAC'$, but $\Delta ABC \ncong \Delta ABC'$.)

Do this at least twice, once if $m\angle BAC$ is an acute angle (i.e., $0° < m\angle BAC < 90°$), and once if $m\angle BAC$ is an obtuse angle (i.e., $90° < m\angle BAC < 180°$).

What happens if $m\angle BAC = 90°$?

[2] A *median* of a triangle is a segment from a vertex of the triangle to the midpoint of the opposite side.

SA 21. *The Euler line.* Draw a triangle. Find the circumcenter C (where the perpendicular bisectors of the sides meet; see SA 6a) and color it blue. Find the centroid D (where the medians meet; see SA 6c) and color it green. Find the orthocenter O (where the altitudes[3] meet) and color it red. Draw a line segment connecting the circumcenter to the orthocenter. What do you notice about the three points C, D, O and the distances between them? State your observation as a conjecture about all triangles. Deform the triangle to convince yourself that your conjecture works. Send me the sketch under the name EULER. All I should see is the original triangle, the colored points, and the line segment.

SA 22. *Pappus' theorem.* Draw two lines ℓ, ℓ' and color them black. Put three points A, B, C on ℓ and three points A', B', C' on ℓ'. Construct the lines (not line segments!)

$$\overleftrightarrow{AB'},\quad \overleftrightarrow{BA'},\quad \overleftrightarrow{AC'},\quad \overleftrightarrow{CA'},\quad \overleftrightarrow{BC'},\quad \overleftrightarrow{CB'}$$

and color them red (to make it easier to distinguish them from ℓ and ℓ'). Then construct the points[4]

$$X = \overleftrightarrow{AB'} \cap \overleftrightarrow{BA'}, \qquad Y = \overleftrightarrow{AC'} \cap \overleftrightarrow{CA'}, \qquad Z = \overleftrightarrow{BC'} \cap \overleftrightarrow{CB'}.$$

a. What do you notice about the points X, Y and Z? State your conclusion as a conjecture.

b. What happens if you move the points A, B, C, A', B', C' around? In particular, does your conjecture still remain true if you change which of A, B, C is between the others?

c. What happens to your sketch when $A = B$? (In particular, what happens to the points X, Y, Z?)

Send me the sketch under the name PAPPUS, with your answers included in a text box.

SA 23. *Ptolemy's theorem.* Draw a circle and inscribe a convex[5] quadrilateral ABCD in it. Measure

$$AB \cdot CD + AD \cdot BC \quad \text{and} \quad AC \cdot BD.$$

a. What do you notice? State your conclusion as a conjecture.

b. Deform the quadrilateral to convince yourself this is a plausible hypothesis. Which deformations maximize the expression $AB \cdot CD + AD \cdot BC$? What kind of quadrilateral gives the maximum value?

c. Is your conjecture still true for an arbitrary quadrilateral (that is, a quadrilateral whose points do not all lie on a common circle)?

Wait, there's more! Measure the area of the quadrilateral $ABCD$, and calculate

$$\Omega = \frac{AC \cdot BD}{\text{area of } ABCD}.$$

Deform the quadrilateral some more and see what happens to the number Ω

d. How can you make Ω as small as possible? How small can it get?

e. How can you make Ω as large as possible? How large can it get?

Send me the sketch under the name PTOLEMY, with your answers included in a text box.

[3] An *altitude* of a triangle is a line through one of the vertices that is perpendicular to the opposite side of the triangle.

[4] The symbol $\cap$ means "intersection". So X is the unique point where $\overleftrightarrow{AB'}$ meets $\overleftrightarrow{BA'}$, etc.

[5] "Convex" just means that if you walk around the circle, you should arrive at A, B, C, and D in that order. So the sides of the quadrilateral are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, and its diagonals are $\overline{AC}$ and $\overline{BD}$.

SA 24. Construct an equilateral triangle ΔABC and measure its area. Construct points D, E, F (outside the triangle) such that

$$A \text{ is the midpoint of } \overline{BD},\ B \text{ is the midpoint of } \overline{CE}, \text{ and } C \text{ is the midpoint of } \overline{AF}.$$

- Use Sketchpad to conjecture a relationship between the area of ΔDEF and the area of ΔABC. Move the sketch around to convince yourself that the construction is correct.
- Prove your conjecture using the axioms and theorems we know about area (no trigonometry).
- Investigate the case that ΔABC is an arbitrary triangle, rather than an equilateral triangle. Does the same pattern seem to occur? Does your proof still work?

4 Transformational geometry

TG 1. Which isometries of the plane are symmetries of. . .

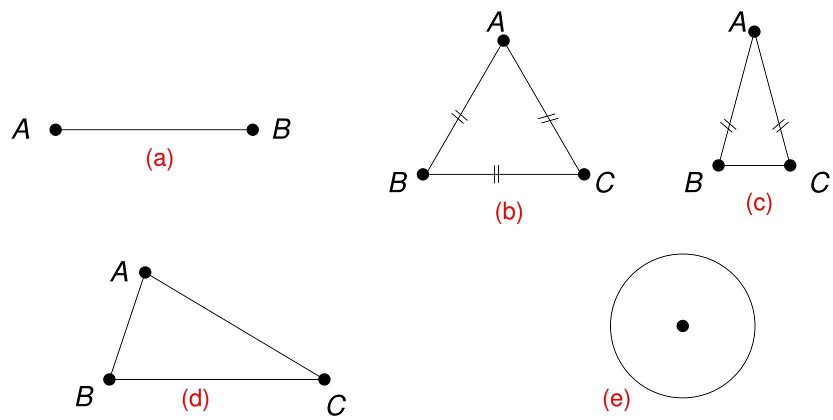
(a) . . . a line segment $\overline{AB}$?

(b) . . . an equilateral triangle ΔABC?

(c) . . . an isosceles triangle ΔABC that is not equilateral? (I.e., $AB = AC$ but $AB \neq BC$.)

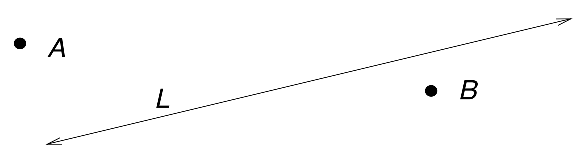
(d) . . . a scalene triangle? (That is, a triangle with three unequal sides.)

(e) . . . a circle with center A?



Make your answers specific (for example, rather than saying "figure X has rotation symmetry," give the possible centers and angles of rotations that are symmetries of X).

TG 2. Consider the following figure.



Which isometries of the plane are symmetries of. . .

(a) . . . point A?

(b) . . . the set consisting of the two points $\{A, B\}$? (That is, the transformations that either fix both A and B, or else interchange them.)

(c) . . . the line L?

(d) . . . the set containing line L and point A?

(e) . . . the set containing line L and both points?

TG 3. Let ℓ be the line with equation $y = 0$ and let m be the line with equation $y = 4$. Describe the transformations $r_\ell \circ r_m$ and $r_m \circ r_\ell$ as simply as possible.

TG 4. Let ℓ be the line with equation $y = 0$ and let m be the line with equation $x = 0$. Describe the transformations $r_\ell \circ r_m$ and $r_m \circ r_\ell$ as simply as possible.

TG 5. Let $\phi : S \to S$ and $\psi : S \to S$ be transformations. Prove that the composition $\phi \circ \psi$ is a transformation by showing that it is 1-1 and onto. (Hint: In order to show that it is 1-1, suppose that it isn't, i.e., there exist distinct elements $x, y \in S$ such that $(\phi \circ \psi)(x) = (\phi \circ \psi)(y)$. Why is this impossible?)

TG 6. (a) Does the set of all translations of the plane form a group? (That is, the set of all transformations $\tau_{\vec{v}}$, where $\vec{v}$ is any vector.) Why or why not?

(b) Does the set of all reflections of the plane form a group? (That is, the set of all transformations r_ℓ, where ℓ is any line.) Why or why not?

TG 7. Consider the transformation of $\mathbb{R}^2$ that takes (x, y) to $(x, 2y)$. Prove that it's affine, but not a similarity and not an isometry.

TG 8. Let m, n be distinct parallel lines. Show that $r_m \circ r_n$ is a translation. What translation is it? That is, describe a vector $\vec{v}$ such that $r_m \circ r_n = \tau_{\vec{v}}$. (Hint: Pick three non-collinear points that are easy to work with, and use the Three-Point Theorem.)

TG 9. Let ℓ be a line and let A be a point on ℓ.

(a) Find a line m such that $r_\ell \circ \rho_{A,90} = r_m$. (Hint: Carry out the composition $r_\ell \circ \rho_{A,90}$ using a piece of paper and/or a transparency. Use your example to make an educated guess about what m is. Then prove that you are correct using the Three-Point Theorem.)

(b) Do the same thing for these other three compositions (that is, express each one of them as reflection across an appropriate line):

$$r_\ell \circ \rho_{A,270}, \qquad \rho_{A,90} \circ r_\ell, \qquad \rho_{A,270} \circ r_\ell.$$

(c) What do you notice about these four transformations?

TG 10. Start by downloading the Geometer's Sketchpad worksheet from
`http://www.math.ku.edu/~jmartin/math409/congruent-triangles.gsp`.

When you open the worksheet, you will see two congruent triangles $T = \Delta ABC$ and $T^* = \Delta A^*B^*C^*$. Try moving one of the vertices around, and observe that the other vertices get dragged around so that the two triangles always remain congruent.

The Three-Point Theorem says that there exists an isometry ψ that takes A to A^*, B to B^*, and C to C^*. The Three-Reflection Theorem says that ψ can be realized as the composition of at most three reflections, i.e., either

$$\psi = \text{id}, \qquad \psi = r_\ell, \qquad \psi = r_\ell \circ r_m, \qquad \text{or} \qquad \psi = r_\ell \circ r_m \circ r_n$$

where ℓ, m, n are appropriately chosen lines.

Before you go any further, think about how you could perform three reflections in a row that would take A, B, C to A^*, B^*, C^* respectively in the sketch you've been given. If you see how to do it, then work through the steps below and try to connect the procedure (and the notation) with your intuition. If you don't see how to do it, then work through the steps below and try to understand why they work.

Step 1: come up with a line ℓ such that $r_\ell(A) = A^*$. Construct this line in Sketchpad (by using an appropriate tool from the **Construct** menu). Then, construct the triangle $T_1 = r_\ell[T]$; that is, actually reflect T across ℓ in your sketch.[6] To help keep track of which object is which, color both the line ℓ and the triangle T_1 red.

Try dragging some of the points around. If you've done the first step correctly, then the reflection line ℓ and the triangle T_1 will change, but $r_\ell(A)$ should always coincide with A^*.

If $T_1 = T^*$. then $r_\ell = \psi$ (by the Three-Point Theorem), so you'd be all done. Unless you drag the points around very carefully, this probably won't be the case In other words, most isometries are not single reflections.

Okay, now you've gotten A moved to A^*. Let $B_1 = r_\ell(B)$, $C_1 = r_\ell(C)$.

Step 2: Come up with a line m such that $r_m(A^*) = r_m(A^*)$ and $r_m(B_1) = B^*$. As before, construct this line and the triangle $T_2 = r_m[T_1]$ in Sketchpad. (Color them blue so you can keep track of them.) Notice that the composition $r_m \circ r_\ell$ moves A and B to A^* and B^* respectively: that is,

$$(r_m \circ r_\ell)(A) = r_m(r_\ell(A)) = r_m(A^*) = A^* \qquad \text{and} \qquad (r_m \circ r_\ell)(B) = r_m(r_\ell(B)) = r_m(B_1) = B^*.$$

In other words, if you first apply r_ℓ and then apply r_m, the two points A and B will get moved to where they're supposed to be, namely to A^* and B^* respectively.

Again, try dragging some of the points around, and make sure that the equalities $(r_m \circ r_\ell)(A) = A^*$ and $(r_m \circ r_\ell)(B) = B^*$ remain true even while m and T_2 move around.

If $\psi = r_m \circ r_\ell$ then you're done. In the example sketch, it doesn't so you're not. At this point we've moved A to A^* and B to B^*, but C has moved to a point $C_2 = r_m(C_1) = (r_m \circ r_\ell)(C)$, which need not equal C^*.

Step 3: Come up with a line n such that

$$r_n(A^*) = A^*, \qquad r_n(B^*) = B^*, \qquad r_n(C_2) = C^*.$$

That is,

$$(r_n \circ r_m \circ r_\ell)(A) = A^*, \qquad (r_n \circ r_m \circ r_\ell)(B) = C^*, \qquad (r_n \circ r_m \circ r_\ell)(B) = C^*,$$

Construct this line n (in green) on your sketch. Without actually performing the reflection, you should be able to see that $r_n[T_2] = T^*$, which is equivalent to the preceding three equations.

Now you're done — you've constructed a transformation $r_n \circ r_m \circ r_\ell$ that takes A, B, C to A^*, B^*, C^*. By the Three-Point Theorem, it must be the case that $r_n \circ r_m \circ r_\ell = \psi$.

Send me the sketch under the name REFLECT. All I should see is: the original triangles $T = \Delta ABC$ and $T^* = \Delta A^*B^*C^*$; the lines ℓ, m and n that you've constructed; and the triangles T_1 and T_2.

[6] To perform a reflection in *Sketchpad*, first highlight (only) the line across which you want to reflect — in this case, ℓ — and select "Mark Mirror" from the **Transform** menu. Then highlight the object(s) you want to reflect across the mirror and select "Reflect".

TG 11. In TG 10, you needed three reflections to transform T to T^*. What kinds of triangles require only two reflections? In other words, suppose I tell you that $\Delta ABC = \Delta A^*B^*C^*$, and that you can't get one triangle from the other by fewer than two reflections. How can you tell whether two or three reflections are required? (Hint: In your sketch for TG 10, move A, B, C around and observe the different kinds of triangles that occur as T_2 and T^*.)

TG 12. Let ℓ be a line, let Q be a point on ℓ, and let θ be any angle. Express $r_\ell \circ \rho_{Q,\theta} \circ r_\ell$ as a single transformation. Express $\rho_{Q,\theta} \circ r_\ell \circ \rho_{Q,\theta}$ as a single transformation.

TG 13. Let X be a point in the plane. Let G be the following set (actually, group) of transformations:

$$G = \{\mathrm{id},\ \rho_{X,120},\ \rho_{X,240}\}.$$

Construct a figure whose symmetries are exactly the transformations in G. (Be careful—you need to make sure that the figure has no reflection symmetries. For example, an equilateral triangle will not work.)

TG 14. Suppose that lines ℓ and m meet at an angle of 40°. Let ϕ be any transformation that can be obtained by repeatedly composing r_ℓ and r_m; for example,

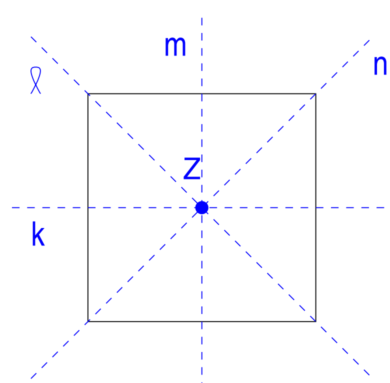
$$\phi = r_m \quad \text{or} \quad \phi = r_\ell \circ r_m \circ r_\ell \quad \text{or} \quad \phi = r_\ell \circ r_m \circ r_\ell \circ r_m \circ r_\ell \circ r_m \circ r_\ell \circ r_m.$$

How many different possibilities are there for ϕ? (The answer is a finite number.)

TG 15. Recall that the group of symmetries of a square S is

$$\mathrm{Sym}(S) = \left\{\mathrm{id},\ \rho_{Z,90},\ \rho_{Z,180},\ \rho_{Z,270},\ r_k,\ r_\ell,\ r_m,\ r_n\right\}$$

where Z is the center of the square, k and ℓ are the two diagonals, and m and n are the lines joining the midpoints of opposite sides, as shown.



(a) Write out the multiplication table for the group of isometries of the plane that are symmetries of S.

(b) A *subgroup* of Sym(S) is defined as a subset of Sym(S) that is itself a group. One example of a subgroup is the set of rotations in S, namely

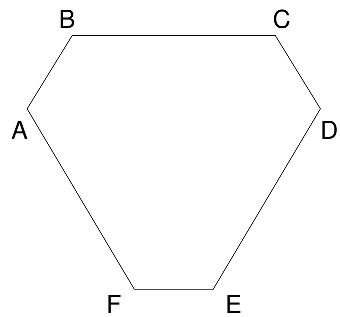
$$\left\{\mathrm{id},\ \rho_{Z,90},\ \rho_{Z,180},\ \rho_{Z,270}\right\}.$$

Another (silly-looking) example of a subgroup is the set

$$\{\mathrm{id}\}$$

(which is, after all, a group, albeit not a very exciting one). Explain how you can use the multiplication table to find other subgroups, and find as many as you can. (In fact, $\mathrm{Sym}(S)$ has a grand total of ten subgroups, including $\mathrm{Sym}(S)$ itself and the two examples above.)

TG 16. Let $H = ABCDEF$ be a hexagon in which all internal angles are $120°$, and $AB = CD = EF$ and $BC = DE = FA$, but $AB \neq BC$, as shown.

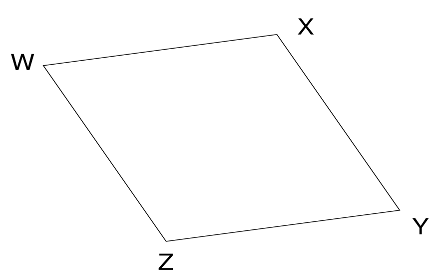


(a) List all the elements of the group $\mathrm{Sym}(H)$. (You don't have to write out the multiplication table.)

(b) How does $\mathrm{Sym}(H)$ compare to the group of symmetries of an equilateral triangle? Explain your answer by drawing a picture.

TG 17. Let Q be a cube in $\mathbb{R}^3$. How many isometries of $\mathbb{R}^3$ are symmetries of Q? (Hint: In §6 of "Notes on transformational geometry," we showed that a regular polygon in $\mathbb{R}^2$ with n sides has $2n$ isometries. Formulate a similar argument here.)

TG 18. Let $\mathcal{R} = WXYZ$ be a rhombus that is not a square.



We already know that $\mathcal{R}$ has exactly four symmetries (see §6.4 of the notes on transformations).

(a) Write out the permutation words for all four symmetries of $\mathcal{R}$.

(b) Compare your results with the list of permutation words for the symmetries of a rectangle (see Example 4 in the notes). What do you notice? Can you explain what is going on?

TG 19. Let $n \geq 3$, and let $\mathcal{P}$ be a polygon with n vertices. Why can't $\mathcal{P}$ have more than $2n$ symmetries? (You may be tempted to say, "Regular polygons have the most possible symmetries, and they have only $2n$ symmetries, so the answer is no." However, that argument is tautological — how do you know that a regular n-gon has the most symmetries of any n-sided polygon? Instead, try counting all possible symmetries of $\mathcal{P}$.)

TG 20. Let $\mathcal{P}$ be a regular n-sided polygon.

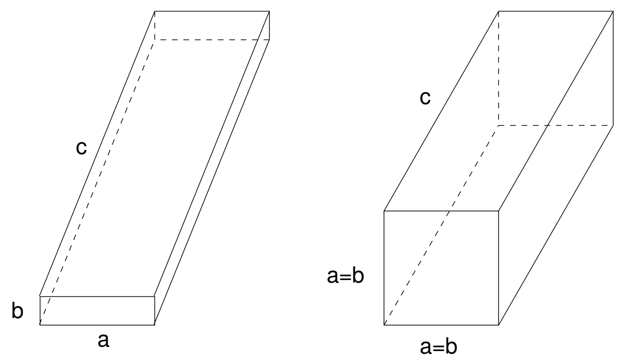
(a.) Describe the permutation words for all $2n$ symmetries of $\mathcal{P}$. (You don't have to write out the entire list, because the number of items in the list depends on n. Instead, describe what kinds of permutation words show up in general.)

(b.) Which ones are rotations and which ones are reflections? (To put it another way, how can you tell whether a symmetry of $\mathcal{P}$ is a rotation or a reflection just by looking at its permutation word?)

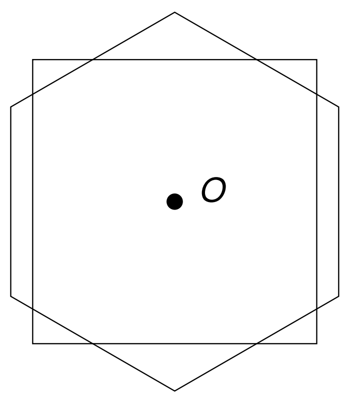
TG 21. Let $\mathcal{P}$ be a rectangular prism in $\mathbb{R}^3$ with length a, width b, and height c.

(a) Suppose that a, b, c, are all different (as in the left-hand figure below). How many symmetries does $\mathcal{P}$ have?

(b) Suppose that $a = b$, but $a \neq c$ and $b \neq c$ (as in the right-hand figure below). How many symmetries does $\mathcal{P}$ have?



TG 22. Let $\mathcal{F}$ be a figure obtained by superimposing a regular n-sided polygon and a regular m-sided polygon with the same center. For example, if $n = 4$ and $m = 6$, then $\mathcal{F}$ might look like this:

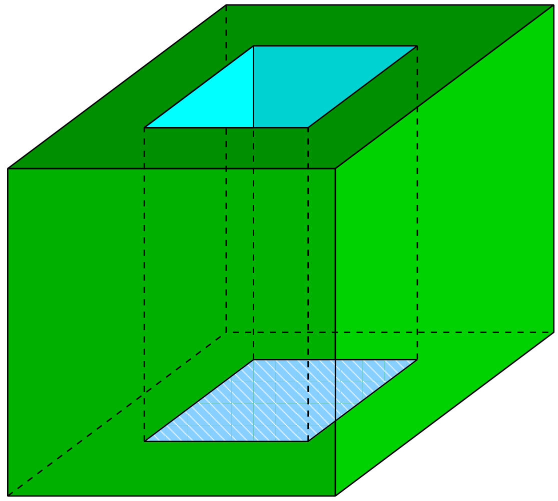


Describe the symmetry group of $\mathcal{F}$.

5 Polyhedra

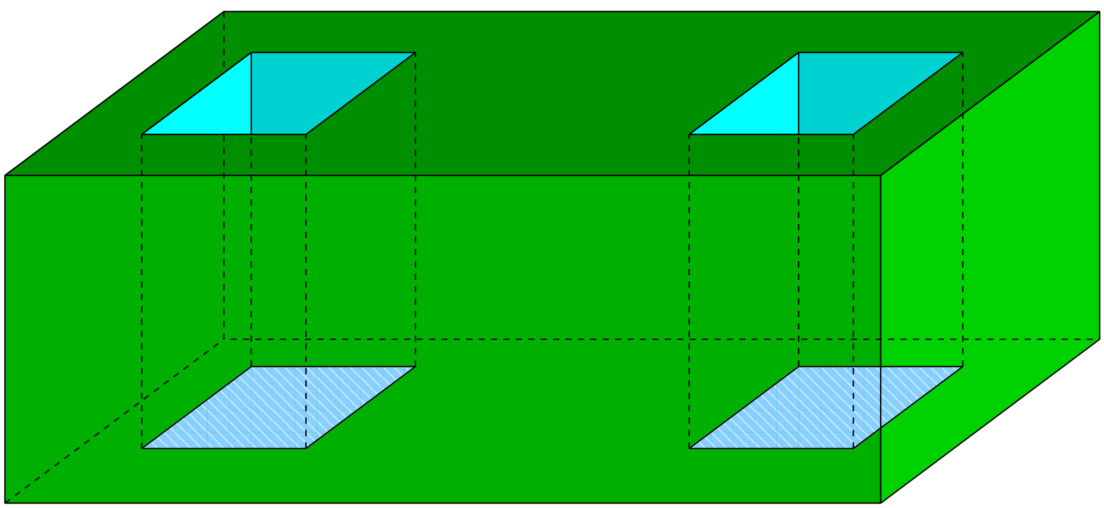
PH 1. Determine v, e and f (the numbers of vertices, edges and faces) for (a) a pyramid whose base is an n-sided polygon ("n-gon"); (b) a bipyramid whose base is an n-gon; (c) a prism whose base is an n-gon. Verify that each of these polyhedra satisfies Euler's formula.

PH 2. Consider a square prism with a hole drilled through the middle (a "big square donut"), as pictured.



(a) Determine v, e and f. Do these numbers satisfy Euler's formula? If not, why not?

(b) Suppose that you drill lots of holes — say h of them. (In the figure below, $h = 2$.) What's the relationship between v, e, f and h?



PH 3. Let $f(n, k)$ be the number of k-dimensional faces of the n-dimensional cube Q_n. (For example, a 3-dimensional cube has 12 edges and 6 faces, so $f(3, 1) = 12$ and $f(3, 2) = 6$.) We saw some of the following values of $f(n, k)$ in class:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	1	0	0	0	0	0
$n=1$	2	1	0	0	0	0
$n=2$	4	4	1	0	0	0
$n=3$	8	12	6	1	0	0
$n=4$	16	32	24	8	1	0
$n=5$	32	80	80	40	10	1

(a) What's the relationship between these numbers and the polynomial $f(x) = 2x+1$? (Hint: Look at $f(x)^2$, then go from there.)

(b) How many constituent pieces (i.e., vertices, edges, faces, etc.) do you think an n-dimensional cube has? (For instance, the square Q_2 has 4 vertices + 4 edges + 1 square = 9 pieces total.)

(c) [Extra credit] What's the connection between your answers to (a) and to (b)?

PH 4. In class and in the notes, we proved that there are only five Platonic (regular) solids in 3-space. However, you might be bothered by an omission in the proof: we didn't actually prove that the sum of angles meeting at a vertex in a polyhedron must be less than 360°. Here's a way to get around that problem that only uses counting.

Suppose that $\mathcal{P}$ is a Platonic solid with v vertices, e edges and f faces. We know that every face is a regular polygon with the same number of sides; as in the notes, we'll call this number s. Also, let's let d be the degree of each vertex (remember, this means the number of edges[7] incident to it; since $\mathcal{P}$ is regular, this is the same for all vertices).

So we have five parameters to work with: v, e, f, s, d. We also know that

$$dv = 2e \tag{1}$$

(by the Handshaking Theorem, because every vertex has degree d) and similarly that

$$sf = 2e. \tag{2}$$

(a) Use Euler's formula and equations (1) and (2) to solve for v in terms of d and s.

(b) By Theorem 1 from the notes on polyhedra, we know that $s \geq 3$ and $d \geq 3$. What does Theorem 2 say about how *large* s and d can be?

(c) At this point, you have reduced the possibilities for s and d to a finite number of cases. In each case, what value do you get for v?

(You should end up with five viable cases, corresponding to the Platonic solids. However, your argument should NOT use what we know about the Platonic solids. The point of the problem is to prove that classification in a different way, so using the classification itself would be a circular argument.)

[7] In fact this is the same as the number of *faces* containing each vertex, but for this argument it's more helpful to think in terms of *edges* containing a vertex.