Plane geometry via axioms: an introduction

Adapted from Judy Roitman

We have kept asking: why do our geometric constructions work? That is, how do we know we are constructing the objects we want to construct?

So far, when we have tried to answer this question, we have appealed to various facts that seemed self-evident. But we can imagine someone challenging us: how do you know that this or that is true?

Why should we believe our mathematical conclusions? The answer to this question necessarily involves some notion of convincing argument. The ancient Greeks were perhaps the first to have the notion of proving things from a minimal set of assumptions, taking as little as possible for granted. The basic way they worked (and the basic way we work when we do mathematics) goes as follows.

First you have some **undefined notions**—you have to start somewhere.

Then you have some basic facts about these notions, called **axioms**. Again, you have to start somewhere.

Then you add **definitions**, which say what new words mean.

At some point you can prove **theorems**, which are statements that logically follow from what you've already done.

And you add more definitions and more theorems, and less important theorems called lemmas and corollaries, and so on until you've built up the enormous edifice we call mathematics. Which is still expanding, because every theorem you prove leads to more questions which lead to more definitions and theorems which lead to more questions, and so on. It's important to know the difference between undefined notions and axioms—things that don't have to be defined or proved—and theorems, which do have to be proved.

For example, we won't define the notion of distance. We will define a circle as the set of points equidistant from a given point (called the center). Note that this was really two definitions: circle, center.

We don't prove that a circle is the set of points equidistant from a given point. You can't prove it. It's a definition. But you can prove that if AB is a diameter of a circle and D is a point on the circle, then ADB is a right angle (see homework problem SA4). In fact you have to prove it, otherwise you don't know whether it's true or not. Then it becomes a theorem and you can use it in proving subsequent theorems.

These distinctions were explicitly made by Euclid, who codified ancient Greek mathematical knowledge in his *Elements*, around the third century BCE. We tend to think of Euclid's *Elements* as being just about geometry, but in fact it included all of mathematics; geometry was only one part.

Euclid's idea was to have a very small set of undefined terms like point and line (i.e., "infinite straight line") — and a small set of axioms:

1. Any two distinct points determine a unique line.
2. Lines can be extended infinitely.
3. Every segment can be made into the radius of a circle.
4. All right angles are equal.
5. The parallel postulate: if two lines L_1, L_2 meet a third line P so that the sum of the inner angles on one side of P is less than $180°$, then L_1 and L_2 can't be parallel.

This notion of proving everything from a small set of basic principles had a powerful effect, ultimately shaping every area of mathematics. Indeed, the smaller the set of basic principles, the better a mathematical system is. The goal is to derive the most powerful rules possible from the simplest building blocks.

To the modern mind Euclid's axiom system is incomplete. He clearly was using a lot of physical intuition that was not made explicit in his axioms, and also suffered from quite a bit of vagueness. For example, he defined a point as "that which has no part", and an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line."[1] This is circular—if you don't already know what an angle is you can't possibly understand it.

Still, Euclid's idea of stating your axioms explicitly and then deriving theorems from them was the birth of modern mathematics as we know it.

The fifth axiom is a famous story. It's clearly a much more complicated statement than the first four. This made a lot of geometers uncomfortable — after all, axioms are supposed to be simple principles that sensible people should all agree are obvious, and it takes a fair amount of effort even to figure out what that postulate is saying (let along whether it is "obvious"). Over the centuries, mathematicians spent a huge amount of effort trying to show that Euclid's fifth axiom could be proven from the first four. It turns out that it can't! More on this later.

[1] From the translation at http://www.perseus.tufts.edu/hopper/text?doc=Euc.+1.