Math 223 Test #2 (11/9/12)
Solutions

(#1a) $\mathbf{v}(t) = \mathbf{i} + 2t^2\mathbf{j} + 2t\mathbf{k}$; speed $= \|\mathbf{v}(t)\| = \sqrt{1+4t^4+4t^2} = \boxed{1+2t^2}$

(#1b) Total distance traveled $= \int_0^3 \|\mathbf{v}\| dt = \displaystyle\int_0^3 (1+2t^2)\, dt \;=\; 2t + \frac{2t^2}{3}\Big|_0^3 \;=\; 3+6 = \boxed{9}$

(#1c) $\mathbf{a}(t) = 4t\mathbf{j}+2\mathbf{k}$; $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t^2 & 2t \\ 0 & 4t & 2 \end{vmatrix} = -4t^2\mathbf{i} - 2\mathbf{j} + 4t\mathbf{k}$; $\|\mathbf{v}\times\mathbf{a}\| = \sqrt{16t^4+4+16t^2} = 2(1+2t^2)$

$$\kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|\|\mathbf{v}\|^3}{=} \tfrac{2(1+2t^2)}{(1+2t^2)^3} = \boxed{\frac{2}{(1+2t^2)^2}}$$

(#1d) $\mathbf{a}'(t) = 4\mathbf{j}$; $(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}' = -8\mathbf{j}$; $\tau = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{\|\mathbf{v}\times\mathbf{a}\|^2} = \dfrac{-8}{(2(1+2t^2))^2} = \boxed{\dfrac{-2}{(1+2t^2)^2}}$

(#1e) Curvature and torsion are both very close to zero when $t = 100$, which indicates that the trajectory is close to a straight line.

(#2b) A watermelon placed at $(1,3)$ will tend to rotate clockwise in the xy-plane; this indicates that $(\nabla\times\mathbf{G})(1,-3)$ should be some negative constant times $\mathbf{k}$.

(#2c) $\operatorname{div}\mathbf{F}(0,0) > 0$, since $(0,0)$ is a source for $\mathbf{F}$ — the arrows all point away from it.

(#2d) $\mathbf{G}$ cannot be a gradient vector field. As seen in (b), its curl is nonzero, so it is not irrotational — but gradient fields must be irrotational (because $\nabla\mathrm{x}(\nabla f) = \mathbf{0}$). Alternately, gradient fields cannot have closed flow lines (because the potential function would increase around a closed curve, which is impossible).

(#3) Note first that $\mathbf{x}'(t) = (-r\sin t,\ r\cos t,\ 1)$ and $\|\mathbf{x}'(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t + 1} = \sqrt{r^2+1}$.

(#3a) $f(\mathbf{x}(t)) = r^2\sin^2 t + r^2\cos^2 t + t^2 = r^2 + t^2$, so

$$\int_C f\, ds \;=\; \int_0^{2\pi} f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\, dt \;=\; \sqrt{r^2+1}\int_0^{2\pi}(r^2+t^2)\, dt \;=\; \sqrt{r^2+1}(r^2t + t^3/3)\Big|_0^{2\pi}$$
$$= \boxed{\sqrt{r^2+1}(2\pi r^2 + 8\pi^3/3)}$$

(#3b) $\mathbf{F}(\mathbf{x}(t)) = (r\sin t,\ -r\cos t,\ 1)$, so

$$\int_C \mathbf{F}\cdot d\mathbf{s} \;=\; \int_0^{2\pi}\mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\, dt \;=\; \int_0^{2\pi}(r\sin t,\ -r\cos t,\ 1)\cdot(-r\sin t,\ r\cos t,\ 1)\, dt$$
$$= \int_0^{2\pi}(-r^2\sin^2 t - r^2\cos^2 t + 1)\, dt \;=\; \int_0^{2\pi}(-r^2+1)\, dt \;=\; \boxed{2\pi(1-r^2)}$$

(#3c) Reversing the orientation of C would not change the scalar line integral in (a), but it would reverse the sign of the vector line integral in (b).

(#4) By Green's theorem, the integral equals the area enclosed by the curve, which is 10.

(#5) We need to find a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = 6x^2 + 2x/y - 4y/x^2$ and $\frac{\partial f}{\partial y} = 4/x - x^2/y^2 + 3$. That is,

$$f = \int (6x^2 + 2x/y - 4y/x^2)\ dx \ = \ 2x^3 + x^2/y + 4y/x + \alpha(y),$$

$$f = \int (4/x - x^2/y^2 + 3\ dy \ = \ 4y/x + x^2/y + 3y + \beta(x)$$

The expression $x^2/y + 4y/x$ occurs on both lines, and looking at the other pieces we must have $\alpha(y) = 3y$ and $\beta(x) = 2x^3$. We conclude that the desired scalar potential function is

$$\boxed{4y/x + x^2/y + 3y + 2x^3.}$$

(#6) There are many answers possible. Full credit was awarded for any vector field whose curl was nonzero. (If $\mathbf{G}$ has a potential function f, then $\nabla f = \mathbf{G}$, and $\nabla \times \mathbf{G} = \nabla \times (\nabla f) = 0$. Therefore, if $\nabla \times \mathbf{G} \neq \mathbf{0}$ then $\mathbf{G}$ cannot have a scalar potential function and you get to go free.)