[2.5] #20: Let $\mathbf{f}(x) = (x^2,\ \cos 3x,\ \ln x)$ and let $g(s,t,u) = s + t^2 + u^3$. Calculate $D(\mathbf{f} \circ g)$ both (a) by evaluating $\mathbf{f} \circ g$ and (b) by using the Chain Rule.

(a) For short, let $\mathbf{h} = \mathbf{f} \circ g$. Then

$$\begin{aligned} h(s,t,u) &= \mathbf{f}(g(s,t,u)) = \mathbf{f}(s + t^2 + u^3) \\ &= \big((s+t^2+u^3)^2,\ \cos(3(s+t^2+u^3)),\ \ln(s+t^2+u^3)\big) \\ &= (h_1(s,t,u),\ h_2(s,t,u),\ h_3(s,t,u)) \end{aligned}$$

$$Dh(s,t,u) = \begin{bmatrix} \dfrac{\partial h_1}{\partial s} & \dfrac{\partial h_1}{\partial t} & \dfrac{\partial h_1}{\partial u} \\ \dfrac{\partial h_2}{\partial s} & \dfrac{\partial h_2}{\partial t} & \dfrac{\partial h_2}{\partial u} \\ \dfrac{\partial h_3}{\partial s} & \dfrac{\partial h_3}{\partial t} & \dfrac{\partial h_3}{\partial u} \end{bmatrix} = \begin{bmatrix} 2(s+t^2+u^3) & 4t(s+t^2+u^3) & 6u^2(s+t^2+u^3) \\ -3\sin(s+t+u^3) & -6t\sin(s+t+u^3) & -9u^2\sin(s+t+u^3) \\ \dfrac{1}{s+t^2+u^3} & \dfrac{2u}{s+t^2+u^3} & \dfrac{3u^2}{s+t^2+u^3} \end{bmatrix}$$

(b) The derivative matrices are

$$D\mathbf{f}(x) = \begin{bmatrix} 2x \\ -3\sin 3x \\ 1/x \end{bmatrix}, \quad Dg(s,t,u) = \begin{bmatrix} 1 & 2t & 3u^2 \end{bmatrix}, \quad D\mathbf{f}(g(s,t,u)) = \begin{bmatrix} 2(s+t^2+u^3) \\ -3\sin(3(s+t^2+u^3)) \\ 1/(s+t^2+u^3) \end{bmatrix}$$

and so the Chain Rule says that

$$D(\mathbf{f} \circ g)(s,t,u) = \begin{bmatrix} 2(s+t^2+u^3) \\ -3\sin(3(s+t^2+u^3)) \\ 1/(s+t^2+u^3) \end{bmatrix} \begin{bmatrix} 1 & 2t & 3u^2 \end{bmatrix}$$

which gives the same result as before. Remember how matrix multiplication works: here we are multiplying a 3×1 matrix M by a 1×3 matrix N, so the result is a 3×3 matrix whose entry in row i and column j is the dot product of the i^{th} row of M with the j^{th} column of N. In this case, the rows of M and the columns of N happen to only have one element each, so their dot product is just their product.

Warning: Be careful here, because $\mathbf{f} \circ g$ and $g \circ \mathbf{f}$ are both well-defined functions. They have different domains and ranges, though: $g \circ \mathbf{f}$ is a function $\mathbb{R} \to \mathbb{R}$ rather than $\mathbb{R}^3 \to \mathbb{R}^3$. Specifically,

$$g \circ \mathbf{f}(x) = g(x^2,\ \cos 3x,\ \ln x) = x^2 + \cos^2(3x) + (\ln x)^3.$$

and so its derivative is

$$(g \circ \mathbf{f})'(x) = 2x - 2\cos 3x \sin 3x + 3(\ln x)^2/x.$$

The Chain Rule works here too. We have already calculated $D\mathbf{f}(x)$, and

$$Dg(\mathbf{f}(x)) = Dg(x^2,\ \cos 3x,\ \ln x) = \begin{bmatrix} 1 & 2\cos 3x & 3(\ln x)^2 \end{bmatrix}$$

and so

$$\begin{aligned} D(g \circ \mathbf{f})(x) &= [Dg(\mathbf{f}(x))][D\mathbf{f}(x)] \\ &= \begin{bmatrix} 1 & 2\cos 3x & 3(\ln x)^2 \end{bmatrix} \begin{bmatrix} 2x \\ -3\sin 3x \\ 1/x \end{bmatrix} \\ &= 2x - 6\sin 3x \cos 3x + 3(\ln x)^2/x. \end{aligned}$$

[2.6] #17: Find an equation for the tangent plane to the surface given by the equation $ze^y \cos x = 1$ at the point $(\pi, 0, -1)$.

This is a surface in $\mathbb{R}^3$; we can think of it as a level surface of the function $f(x, y, z) = ze^y \cos x - 1$. The gradient $\nabla f(\pi, 0, -1)$ will give a normal vector to the tangent plane, so we calculate

$$\begin{aligned} \nabla f(x, y, z) &= \big(f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z)\big) \\ &= (-ze^y \sin x,\ ze^y \cos x,\ e^y \cos x), \\ \nabla f(\pi, 0, -1) &= (0,\ -1,\ 1). \end{aligned}$$

The tangent plane also passes through the point $(\pi, 0, -1)$, so its equation is

$$0(x - \pi) + (-1)(y - 0) + 1(z - (-1)) = 0 \qquad \text{or more simply} \qquad \boxed{-y + z + 1 = 0.}$$

By contrast, what if we want to find the tangent space to the graph of f? This graph is defined by the equation $w = f(x, y, z)$ in $\mathbb{R}^4$, and the point $(x, y, z, w) = (\pi, 0, -1, 0)$ lies on it. It's important to realize that the graph is a 3-dimensional object (just like the graph of a function $\mathbb{R} \to \mathbb{R}$ is a 1-dimensional object that lives in $\mathbb{R}^2$, and the graph of a function $\mathbb{R}^2 \to \mathbb{R}$ is a 2-dimensional object that lives in $\mathbb{R}^3$), and so its tangent space will be 3-dimensional as well. We can find the equation to the tangent space by the formula

$$\begin{aligned} w - 0 &= f_x(\pi, 0, -1)(x - \pi) + f_y(\pi, 0, -1)(y - 0) + f_z(\pi, 0, -1)(z + 1) \\ w &= 0(x - \pi) - (y - 0) + (z + 1) = -y + z + 1. \end{aligned}$$

[2.6] #21: Calculate the plane tangent to the surface $x \sin y + xz^2 = 2e^{yz}$ at the point $(2, \pi/2, 0)$ in two ways.

(a) We can solve for x in terms of the other two variables as

$$x = \frac{2e^{yz}}{\sin y + z^2}$$

and we can therefore regard x as a function of y and z; call it $x = p(y, z)$. So $p(\pi/2, 0) = 2$, and we can think of the surface as the graph of p in $\mathbb{R}^3$; we just need to remember that x is the dependent variable and y, z are independent. Therefore, we can find the equation of the tangent plane by using the formula

$$x = p_y(\pi/2, 0)(y - \pi/2) + p_z(\pi/2, 0)(z - 0) + p(\pi/2, 0). \qquad (*)$$

The differentiation requires the Quotient Rule and is unpleasant; I used a computer to get

$$p_y(y, z) = \frac{2e^{yz}\left(z \sin y + z^3 - \cos y\right)}{(\sin y + z^2)^2}, \qquad p_z(y, z) = \frac{2e^{yz}\left(y \sin y + yz^2 - 2z\right)}{(\sin y + z^2)^2}.$$

The good news is that plugging in $(y, z) = (\pi/2, 0)$ simplifies matters: $p_y(\pi/2, 0) = 0$ and $p_z(y, z) = \pi$. So equation (*) becomes

$$\boxed{x = \pi z + 2.}$$

(b) Now let's think of the surface as a level surface of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x \sin y + xz^2 - 2e^{yz}.$$

Specifically, the surface we are interested in is defined by the equation $f(x, y, z) = 0$. The normal vector we are looking for is $\nabla f(2, \pi/2, 0)$. This differentiation is far less unpleasant:

$$\begin{aligned} \nabla f(x, y, z) &= f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} \\ &= (\sin y + z^2)\mathbf{i} + (x \cos y - 2ze^{yz})\mathbf{j} + (2xz - 2ye^{yz})\mathbf{k} \\ \nabla f(2, \pi/2, 0) &= \mathbf{i} + 0\mathbf{j} - \pi\mathbf{k} \end{aligned}$$

and so the equation of the tangent plane is

$$\begin{aligned} (\nabla f(2, \pi/2, 0)) \cdot (x - 2, y - \pi/2, z - 0) &= 0 \\ (1, 0, -\pi) \cdot (x - 2, y - \pi/2, z - 0) &= 0 \\ x - 2 - \pi z &= 0 \end{aligned}$$

which is equivalent to the equation $x = \pi z + 2$ found by the first method.