Reparametrization

The green exam featured line integrals over the curve C parametrized by $(t, t^3/3, t^2/2)$, for $0 \le t \le 1$. The solution to problem #1g was

$$\begin{aligned}
\int_C (3xy + 2z + 1)^{1/2}\, ds &= \int_0^1 (t^4 + t^2 + 1)^{1/2} \left\| \frac{d\mathbf{x}}{dt} \right\| dt \\
&= \int_0^1 (t^4 + t^2 + 1)^{1/2} (t^4 + t^2 + 1)^{1/2}\, dt \\
&= \int_0^1 (t^4 + t^2 + 1)\, dt \\
&= \left(\frac{t^5}{5} + \frac{t^3}{3} + t \right) \Bigg|_0^1 = \frac{1}{5} + \frac{1}{3} + 1 = \frac{23}{15}.
\end{aligned}$$

and the solution to problem #1h was

$$\begin{aligned}
\int_C z\, dx + y\, dy + x\, dz &= \int_0^1 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt \\
&= \int_0^1 (t^2/2, t^3/3, t) \cdot (1, t^2, t)\, dt \\
&= \int_0^1 3t^2/2 + t^5/3\, dt \\
&= \left(\frac{t^3}{2} + \frac{t^6}{18} \right) \Bigg|_0^1 = \frac{1}{2} + \frac{1}{18} = \frac{5}{9}.
\end{aligned}$$

(On the blue version, x and z were interchanged; the integrals worked out the same way.)

Reversing the starting and ending points would require a reparametrization such as

$$\mathbf{y}(t) = \left(1 - t,\ \frac{(1-t)^3}{3},\ \frac{(1-t)^2}{2} \right), \qquad 0 \le t \le 1.$$

(Note that now $\mathbf{y}(0) = \mathbf{x}(1) = (1, 1, 1)$ and $\mathbf{y}(1) = \mathbf{x}(0) = (0, 0, 0)$.)

If you work out the integrals, you will find that

$$\int_{\mathbf{y}} (3xy + 2z + 1)^{1/2}\, ds = \int_{\mathbf{x}} (3xy + 2z + 1)^{1/2}\, ds$$

and

$$\int_{\mathbf{y}} z\, dx + y\, dy + x\, dz = -\int_{\mathbf{x}} z\, dx + y\, dy + x\, dz.$$

This is consistent with the principle that an orientation-reversing reorientation of a curve *preserves* the sign of *scalar* line integrals along it, but **reverses** the sign of **vector** line integrals (Theorem 1.5, p. 371 of Colley).