Math 223, Fall 2010
Extra Credit Problems for HW #7
Due date: Friday 10/22/10

(#1) You are designing a vector calculus exam, and you would like to include an arclength problem that involves something other than a line segment or a helix. On the other hand, you don't want your students to spend the entire fifty minutes working out one of the repulsive integrals that usually accompany arclength problems. Come up with an interesting curve in $\mathbb{R}^n$ (where $n \geq 3$) whose arclength can actually be calculated.

(#2) Suppose that you have a parametrized curve in 3-space that happens to have the form

$$\mathbf{x}(t) \ = \ (f(t),\ g(t),\ g(t)).$$

(a) Explain verbally why the torsion of such a curve should be zero.

(b) Prove that the torsion is in fact zero for all t.

Hint #1: You can do this by showing that $d\mathbf{B}/ds = 0$. Since you don't know what the functions f and g are, you will really have to use general facts about vectors!)

Hint #2: Another way to do the problem is to first do Exercise 31, which is to prove the formula [stated in class]

$$\tau = \frac{(\mathbf{vxa}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2}.$$

(#3) Consider the hyperplanes in $\mathbb{R}^n$ defined by the set of equations

$$x_1 = x_2, \quad x_1 = x_3, \quad \ldots, \; x_1 = x_n, \quad x_2 = x_3, \quad \ldots, \quad \ldots, \quad x_{n-1} = x_n.$$

For example, if $n = 2$, then we are talking about the single line $y = x$ in $\mathbb{R}^2$.

If $n = 3$, then we are talking about the three planes $y = x$, $z = x$, and $z = y$ in $\mathbb{R}^3$.

In general, we're talking about $\binom{n}{2} = \frac{n(n-1)}{2}$ hyperplanes in $\mathbb{R}^n$, each of which is itself a $(n-1)$-dimensional space.

If we think of these hyperplanes as slicing up space into regions, then, in terms of n, how many regions are there?

For example, if $n = 2$, then the line $y = x$ splits $\mathbb{R}^2$ into two regions (namely, northwest and southeast of the line).

Here are some ways of approaching this problem:

- Start by thinking about the cases $n = 2$ and $n = 3$.
- Conjecture a pattern for general n.
- Figure out why your pattern works for the cases you can observe.
- Try to generalize the cases you can actually see ($n = 2$, $n = 3$, and $n = 1$ – what does that last case mean anyway?) to the cases you can't see directly.