Math 141, Fall 2009
Test #2 Redo Answers

Problem R3b [10 pts]

Evaluate

$$\lim_{x\to\infty} (9^x + 7)^{\left(\frac{1}{2x-5}\right)}.$$

Denote the limit by L. Then

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln\left[(9^x+7)^{\left(\frac{1}{2x-5}\right)}\right] \\
&= \lim_{x\to\infty} \frac{1}{2x-5}\ln(9^x+7) \\
&= \lim_{x\to\infty} \frac{\ln(9^x+7)}{2x-5} && (\infty/\infty \text{ form}) \\
&= \lim_{x\to\infty} \frac{\frac{1}{9^x+7}(\ln 9)(9^x)}{2} && (\text{by LHR}) \\
&= \frac{\ln 9}{2}\cdot \lim_{x\to\infty}\frac{9^x}{9^x+7} && (*) \\
&= \frac{\ln 9}{2}\cdot \lim_{x\to\infty}\frac{1}{1+7\cdot 9^{-x}} \\
&= \frac{\ln 9}{2}\cdot 1 \;=\; \ln(9^{1/2}) \;=\; \ln 3.
\end{aligned}$$

Therefore,

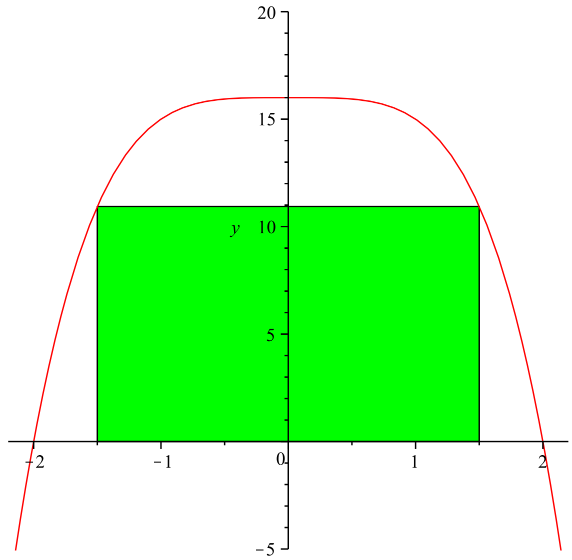
$$\boxed{L = 3.}$$

Note: The limit in step $(*)$ is an ∞/∞ form, so it is also possible to evaluate it using LHR:

$$\lim_{x\to\infty}\frac{9^x}{9^x+7} \;=\; \lim_{x\to\infty}\frac{(\ln 9)9^x}{(\ln 9)9^x} \;=\; 1.$$

Problem R4 [20 pts] Let A be the region lying above the x-axis and under the graph of $f(x) = 16 - x^4$. A rectangle is to be inscribed in A so that one side of the rectangle lies on the x-axis. Find the dimensions (base and height) that maximize the area of the rectangle.

Here is a picture of a rectangle inscribed under the graph:



The y-intercepts of the graph of $f(x)$ are ± 2.

The two upper corners of the rectangle will be $(x, 16 - x^4)$, and $(-x, 16 - x^4)$ for some $x \in [0, 2]$. Therefore:

Base of rectangle	$=$	$2x$
Height of rectangle	$=$	$16 - x^4$
Area	$=$	$A(x) = 2x(16 - x^4) = 32x - 2x^5$.

Next, we find the maximum value of $A(x)$ on $[0, 2]$ by calculating its critical values:

$$\begin{aligned} A'(x) = 32 - 10x^4 &= 0 \\ 10x^4 &= 32 \\ x^4 &= 16/5 \\ x &= 2 \cdot 5^{-1/4}. \end{aligned}$$

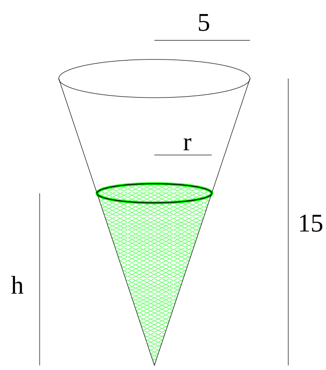
Furthermore, $A''(x) = -40x^3$ is clearly negative for any positive value of x (such as the critical value just calculated). Therefore, by the Second Derivative Test, it is a local maximum. It's the only critical value on $[0, 2]$, so it is the absolute maximum on that interval.

Therefore the dimensions of the maximum-area rectangle are:

$$\boxed{\text{base} = 4 \cdot 5^{-1/4}, \qquad \text{height} = \frac{64}{5} = 12.8.}$$

Problem R5 [20 pts] An ice cream cone is 15 cm high and has diameter 5 cm at its top. The cone is partially full of melted ice cream, which is dripping out of the bottom of the cone at a rate of 3 cm^3/sec. The top surface of the ice cream is a circle that shrinks as the ice cream melts. At what rate is the area of that surface decreasing when there are 60 cm^3 of ice cream left in the cone?

Here is the picture (with green ice cream):



The melted ice cream is in the shape of a cone with radius r and height h. By similar triangles, we have $h/r = 15/2.5 = 6$, so $h = 6r$. Therefore, we can write the volume V and surface area A as functions of r:

$$V = \frac{\pi}{3} r^2 h = 2\pi r^3,$$
$$A = \pi r^2.$$

Note that the problem asks us to find $A' = dA/dt$. Differentiating with respect to time, we get

$$V' = 6\pi r^2 r',$$
$$A' = 2\pi r r'.$$

In particular,

$$A' = \frac{V'}{3r} \tag{$*$}$$

It is given that $V' = -3$ cm^3/sec. Meanwhile, solving $V = 2\pi r^3$ for r gives $r = (V/2\pi)^{1/3}$, so when $V = 60$ cm^3, we have

$$r = (30/\pi)^{1/3} \text{ cm}$$

and therefore

$$A' = \frac{3}{3(30/\pi)^{1/3})} = \boxed{\left(\frac{\pi}{30}\right)^{1/3} \text{ cm}^2/\text{sec}} \approx 0.47135 \text{ cm}^2/\text{sec}.$$