Problem HP7: First, $p(x)/q(x)$ will have a diagonal asymptote if and only if the degree of $p(x)$ is exactly one more than the degree of $q(x)$. That is, for some n, the rational function $f(x)$ has the form

$$f(x) \;=\; \frac{p(x)}{q(x)} \;=\; \frac{\alpha x^{n+1} + \beta x^n + \cdots}{\gamma x^n + \delta x^{n-1} + \cdots}.$$

In this case, $f(x)$ will get closer and closer to a line with slope α/γ as $x \to \pm\infty$.

On the other hand, this doesn't mean that the line $y = (\alpha/\gamma)x$ is the asymptote. Why should the asymptote have to have y-intercept zero? After all, shifting the graph of $f(x)$ up or down should shift the asymptote by the same amount.

To find the equation of the asymptote, we go back to the definition: An asymptote is a line to which the graph of $f(x)$ gets closer and closer as $x \to \pm\infty$. Algebraically, this should mean that

$$\lim_{x\to\pm\infty} \left[\frac{p}{q} - (mx+b)\right] \;=\; 0$$

where $y = mx + b$ is the equation of the asymptote, and I've abbreviated $p = p(x)$, $q = q(x)$. It turns out that m and b are determined by the two leading coefficients of each of p and q, for the following reason. The last equation says that

$$\lim_{x\to\pm\infty} \left[\frac{p-(mx+b)q}{q}\right] \;=\; 0$$

and we know that this will happen if and only if the degree of $p - (mx + b)q$ is strictly less than the degree of q. In other words, the x^{n+1} and x^n coefficients of $p - (mx + b)q$ must vanish. If, as before, we write

$$p \;=\; \alpha x^{n+1} + \beta x^n + \cdots, \qquad q \;=\; \gamma x^n + \delta x^{n-1} + \cdots$$

then

$$\begin{aligned} p - (mx+b)q &= (\alpha x^{n+1} + \beta x^n + \cdots) - (mx+b)(\gamma x^n + \delta x^{n-1} + \cdots) \\ &= (\alpha - m\gamma)x^{n+1} + (\beta - b\gamma - m\delta)x^n + [\text{lower-order terms}]. \end{aligned}$$

Setting these coefficients to zero and solving for m and b in terms of $\alpha, \beta, \gamma, \delta$, we get

$$m = \alpha/\gamma, \qquad b = \frac{\beta\gamma - \alpha\delta}{\gamma^2}$$

and this tells us the equation of the diagonal asymptote.

In fact, this is the same as the quotient upon dividing p by q using polynomial long division. (The remainder doesn't affect the equation of the asymptote.)

I awarded 1 point for noticing the condition $\deg(p) = \deg(q) + 1$ (it wasn't enough just to say that $\deg(p) > \deg(q)$, because in fact if $\deg(p) > \deg(q) + 1$ then $f(x)$ does *not* have a diagonal asymptote); 1 point for figuring out the slope; and 1 point for figuring out how to obtain the equation.