Math 141 Honors Problems #15
Due date: Monday 12/14/09, 4:30 PM, Snow 120

HP29 [**4 points**]

Evaluate the improper integrals

$$\int_1^\infty \frac{\ln x}{x^2}\,dx, \quad \int_1^\infty \frac{(\ln x)^2}{x^2}\,dx, \quad \int_1^\infty \frac{(\ln x)^3}{x^2}\,dx, \quad \int_1^\infty \frac{(\ln x)^4}{x^2}\,dx, \quad \ldots$$

What pattern do you observe?

How might you prove that the pattern continues to hold for all powers of $\ln x$ in the numerator?

What about these integrals?

$$\int_1^\infty \frac{\ln x}{x^3}\,dx, \quad \int_1^\infty \frac{(\ln x)^2}{x^3}\,dx, \quad \int_1^\infty \frac{(\ln x)^3}{x^3}\,dx, \quad \int_1^\infty \frac{(\ln x)^4}{x^3}\,dx, \quad \ldots$$

Or even

$$\int_1^\infty \frac{\ln x}{x^p}\,dx, \quad \int_1^\infty \frac{(\ln x)^2}{x^p}\,dx, \quad \int_1^\infty \frac{(\ln x)^3}{x^p}\,dx, \quad \int_1^\infty \frac{(\ln x)^4}{x^p}\,dx, \quad \ldots$$

where p is an arbitrary positive integer?

HP30 [**4 points**]

A *Lamé curve* is a curve defined by the equation

$$|x|^p + |y|^p = 1$$

for some positive number p. For example, if $p = 2$ then the Lamé curve is the unit circle, and for $p = 1$ it is a diamond (i.e., a square with vertices at $(0, \pm 1)$ and $(\pm 1, 0)$. The larger p is, the "fatter" the curve gets. Lamé curves with $0 < p < 1$ might reasonably be called "astroids" (from the Greek for "star-shaped", although that term is traditionally reserved for the particular case $p = 2/3$. (For an attractive picture of Lamé curves, see `http://mathworld.wolfram.com/Superellipse.html`.)

Use calculus to show some or all of the following facts (arranged in order from easiest to hardest):

- For $p = 2/3$, the area inside the Lamé curve is $3\pi/8$.
- For $p = 1/2$, the area inside the Lamé curve is $2/3$.
- For $p = 2/3$, the perimeter of the Lamé curve is $3\pi/8$.
- For $p = 1/2$, the perimeter of the Lamé curve is $4 + 2\sqrt{2}\ln(1 + \sqrt{2})$.

HP31 [**3 points**]

Design a Math 121/141 final exam problem on the topic of arc length. Keep in mind that calculators are not allowed on the final exam, so you need to design a problem that is not too easy (i.e., "use the arc length formula to find the length of a line segment") but can be evaluated in closed form with nothing but pencil and paper. (No fair borrowing an example from the textbook!)