Math 141 Honors Problems #1
Due date: Tuesday, 8/25/09

HP1 [3 points] As discussed in class on Thursday (and as you may have already known), if you add up the binomial coefficients in the n^{th} row of Pascal's Triangle, you get 2^n. That is,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

What happens if you *square* numbers before adding them? That is, if you define a function $f : \mathbb{N} \to \mathbb{N}$ by the formula

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}^2$$

then is there a simpler formula for $f(n)$? Calculate the value of $f(n)$ for a few small values of n, then try and make a conjecture. (Hint: Look at Pascal's Triangle again.) *Bonus problem* (harder): Once you've found the answer, can you explain why it is true for all values of n?

Note: I know that you can find the formula for $f(n)$ from external sources; for example, it's right there on the Wikipedia page on Pascal's Triangle. Please don't. The point of this problem is for you to find it yourself (which ought to be a lot more fun than just looking it up). I'm not going to base any part of your grade on just writing down the correct formula.

HP2 [3 points] Factorials are hard to calculate: to find the value of $n!$, you need to perform n separate multplications, involving larger and larger numbers. Even using the factorial button on a calculator, this can take a while if n is large (even assuming your calculator can handle numbers as big as $n!$ in the first place; my TI-83 Plus can only compute $n!$ when $n \leq 69$). Fortunately, there is an amazingly good way to *approximate* factorials, called *Stirling's formula*:

$$s(n) = n^n e^{-n} \sqrt{2\pi n}.$$

(a) Calculate the numbers $1!$, $2!$, $\ldots$, $15!$. Then calculate $s(1)$, $s(2)$, $\ldots$, $s(15)$. (You can avoid a certain amount of tedium by using your calculator to define $s(n)$ as a function and making a table of values.)

(b) One way to measure the error of this approximation is to calculate the numbers

$$n! - s(n)$$

for all values of n. (The closer this is to 0, the better the approximation.) Another possibility is to calculate the numbers

$$\frac{n!}{s(n)}$$

for all values of n. (The closer this is to 1, the better the approximation.)

Try calculating both $n! - s(n)$ and $n!/s(n)$ for several values of n (not necessarily all of them, but enough so that you can see a pattern). What is happening as n gets larger and larger? Which of these two methods do you think is a better measurement of the accuracy of the approximation? (There's not necessarily a right answer to this question.)