**Sample Problems for the Final of Math 121**, Fall, 2005

The following is a collection of various types of sample problems covering sections 3.8, 4.1, 4.5, and 4.8–6.5 of the text which constitute only **part of** the common Math 121 Final. Since **the final is comprehensive**, this collection **should be complemented** by the sample problems for the midterm and the real midterm of this semester when preparing for the final. Another file posted separately contains a sample Math 121 Final, which gives a concrete example of a real final and also provides a good set of review problems. Note that the skills needed to solve the Gateway Exam problems are expected from you when you take the final. In particular, you are expected to have memorized the derivatives of the basic functions like $x^a$, $a^x$, $\log_a(x)$, $\sin(x)$, $\sin^{-1}(x)$, $\cos(x)$, $\cos^{-1}(x)$, $\tan(x)$, $\tan^{-1}(x)$, etc. Also note that most of the problems in this list can and may be formulated, in the actual final, **in a form** (as a multiple-choice, true-false, or essay problem) **different** from how they are formulated here. (The following problems are collected/edited from various sources including previous review materials from KU professors and the textbook.)

Except for multiple-choice problems and true-false problems, you have to **show your work** and **find the exact solution**.

1. Find the following limits. (i) $\lim_{x\to 0+} \ln(x+1)\cot(x)$. (ii) $\lim_{x\to\infty}\left(1+\sqrt{5}x\right)^{1/\ln(x)}$.

2. Let $y=\tan^{-1}(x-1)$. (i) Find the differential $dy$. (ii) Evaluate $dy$ and $\Delta y$ if $x=2$ and $\Delta x = dx = 0.01$. (iii) Use $dy$ to estimate $\tan^{-1}(1.01)$, i.e. the $y$-value at $x=2.01$.

3. Evaluate (i) $\displaystyle\int_{-\infty}^{1}\frac{1}{\sqrt{3-x}}dx$ and (ii) $\displaystyle\int_{-2}^{2}\frac{1}{x^6}dx$, if convergent.

4. For $f(x)=\dfrac{\sqrt{x}+x\left(\sqrt{1-x^2}\right)^{-1}}{x}$, find (i) the most general antiderivative of $f$ on the interval $(0,1)$, and (ii) the antiderivative $F$ of $f$ with $F(1/2)=3$.

5. What constant acceleration (in mi/h$^2$) is required to increase the speed of a car from 45 mi/h to 60 mi/h in 4 seconds?

6. If $f''(x)=6x+x^{-2}$, $f(1)=2$, and $f(2)=-1$, find $f$.

7. If $\displaystyle\int_2^4 f(x)\,dx=-1$, $\displaystyle\int_5^7 f(x)\,dx=3$, and $\displaystyle\int_4^7 f(x)\,dx=2$, find $\displaystyle\int_2^5 f(x)\,dx$.

8. Find the area of the region (in the first quadrant) bounded by the $y$-axis and the curve $x=y^{2/3}-y$ (whose $y$-intercepts are 0 and 1).

9. Compute

(a) $\displaystyle\int\frac{2x-1}{x^2+1}dx$, (b) $\displaystyle\int_0^1\frac{2x^3+1}{x^4+2x+1}dx$, (c) $\displaystyle\int\frac{3}{\sqrt{1-x^2}}dx$,

(d) $\displaystyle\int x^4\ln(x)\,dx$, (e) $\displaystyle\int_1^2 xe^{-x+1}dx$, (f) $\displaystyle\int\sin^2(x)\cos^3(x)\,dx$,

(g) $\displaystyle\int\frac{x-2}{x^2-3x-4}dx$.

10. Differentiate $\displaystyle\int_1^{\sqrt{x}}\frac{\cos(t)}{t^2}dt$, and $\displaystyle\int_0^{x^2}\cos^2 t\,dt$.

11. If $f$ is continuous and $\int_3^{24} f(x)\,dx = 8$, find $\int_1^2 x^2 f\left(3x^3\right) dx$.

12. Find the length of the parametric curve $(x, y) = \left(t^3 - \sin(t), \ln(t)\right)$ with $1 \leq t \leq 2$ as a concrete definite integral without actually computing its value.

13. A spring exerts a restoring force proportional to the distance it is stretched from its natural length. We stretch it 2 feet beyond its natural length and measure the restoring force at 120 pounds. How much work, in foot-pounds, is done in stretching the spring an additional 3 feet (thus from 2 feet beyond its natural length to 5 feet beyond its natural length)?

14. A 10 ft ladder is leaning against the wall. If the bottom of the ladder is being pulled away (from the wall) at the constant rate of 2 ft/sec, how fast is the top coming down when the top is 6 ft above the ground?

15. Two cars start moving from the same point. One car travels north at 30 mph and the other car travels east at 40 mph. Let $d(t)$ denote the distance between the cars at time $t$ (the number of hours after the cars leave the initial point). How fast is the distance between the cars increasing two hours later?

16. A man starts walking north at 5 ft/s from point $P$. Ten seconds later a woman starts walking east at $4$ ft/s from a point $100$ ft due east of $P$. At what rate are the people moving apart 3 seconds after the woman starts walking?

17. A tank of the shape of a circular cone with its vertex pointing downward (and its top horizontal) is being filled with water. Assume that the radius of its circular top is 16 m and its height (i.e. the distance from the vertex to the top) is 20 m. Let $h(t)$ be the water level (i.e. the distance from the vertex to the water surface) in the tank at time $t$ in minutes. If the water is being pumped into the tank at the rate of 5 $m^3$/min starting from $t = 0$, how fast is the water level rising at time $t = 10$?

18. Find the derivatives of the functions (i) $F(x) = \int_0^{\sqrt{x}} \sin^{-1}\left(t^2\right) dt$, (ii) $G(x) = \int_x^{x^2} \sqrt{1+t^2}dt$, and (iii) $H(x) = \int_1^{\ln(x)} \tan^{-1}(t)\,dt$.

19. For a function $f$ with a continuous derivative $f'$ on the whole real line, $\lim_{h\to 0} \frac{1}{h}\int_a^{a+h} f'(x)dx =$

(a) $f'(a)$, (b) $f(0)$, (c) $f(a)$, (d) $f'(0)$, (e) none of the above.

20. Find a continuous function $f$ and a constant $a$ such that $x^2 - \int_a^x \sqrt{f(t)}dt = 6x - 9$.

21. (a) Give the iterative formula for Newton's method for approximating a root of an equation $f(x) = 0$, where $f$ is a differentiable function on the real line. (b) Use Newton's method with first guess $x_0 = 2$ to approximate the solution of the equation $x^3 - 2x - 8 = 0$ by listing the first 3 numbers in the sequence of approximations obtained by the Newton's method.

22. Given a function $f$ with a continuous derivative $f'$ on the real line $\mathbb{R}$ and with $f(2) = 0$, $f'(2) = 5$, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to\infty} f'(x) = 3$, evaluate the following limits: (a) $\lim_{x\to 2} \frac{f(x)}{x^2-4}$, (b) $\lim_{x\to 2} \frac{f(x)}{\cos(x)}$, (c) $\lim_{x\to\infty} \frac{\ln(x)}{f(x)}$.

23. A particle moves along the $y$-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = t\cos t$. At time $t = 0$, the position of the particle is $y = 3$. (a) For what intervals of $t$, $0 \leq t \leq 5$, is the particle moving upward? (b) Write an expression for the acceleration $a(t)$ of the particle in terms of $t$. (c) Write an expression for the position $y(t)$ of the particle in terms of $t$. (d) Find the position of the particle at the

moment when its velocity becomes zero for the first time after the beginning of motion.

24. A particle with velocity at any time $t$ given by $v(t) = e^t$ moves in a straight line. How far does the particle move from time $t = 0$ to $t = 2$?

25. In a three hour trip, the velocity of a car at each half hour was recorded as follows:

| Time (Hours) | 0 | .5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| Velocity (MPH) | 0 | 40 | 55 | 50 | 35 | 30 | 0 |

Estimate the distance traveled using the Simpson's approximation $S_6$ and estimate the average velocity of the car during this trip.

26. Express as a concrete sum of numbers the approximation $T_6$ to $\int_0^3 \sqrt{x^2+1}\,dx$ obtained by the Trapezoidal Rule.

27. Evaluate the following integrals:

(a) $\int e^{\sin x} \cos x \; dx$, (b) $\int \frac{x+1}{x^2+2x+5}\,dx$, (c) $\int \frac{1}{(1-4x)^2}\,dx$,

(d) $\int x \sin(x) \; dx$, (e) $\int_0^2 6x(x^2+2)^2\,dx$, (f) $\int x^2\sqrt{1+x^3}\,dx$,

(g) $\int_0^{\pi/2} \cos x\sqrt{\sin x}\,dx$, (h) $\int_4^6 x\sqrt{x-4}\,dx$, (i) $\int_2^3 (x+\frac{1}{x})^2\,dx$,

(j) $\int (xe^x + e^{1+x})\,dx$, (k) $\int \frac{dt}{9t^2+4}$, (l) $\int x(\ln x)\,dx$.

28. Let $F(x) = \int_0^{f(x)} \tan(t)\,dt$ for a differentiable function $f$. Then $F'(x) =$

(a) $\tan(x)$, (b) $\tan(x)\,f'(x)$, (c) $f(\tan(x))$, (d) $\sec^2(x)$, (e) none of the above.

29. If $\int_0^k (2kx - x^2)\,dx = 18$, then $k =$

(a) $-9$, (b) $-3$, (c) 3, (d) 9, (e) none of the above.

30. If the function $g$ has a continuous derivative on $[0, c]$, then $\int_0^c g'(x)\,dx =$

(a) $g(c) - g(0)$, (b) $g(x) + c$, (c) $|g(x) - g(0)|$, (d) $g(c)$, (e) none of the above.

31. Given the following graph of a function $f$, define the function $g(x) = \int_{-5}^x f(t)\,dt$. Determine whether each of the following statements is true or false.

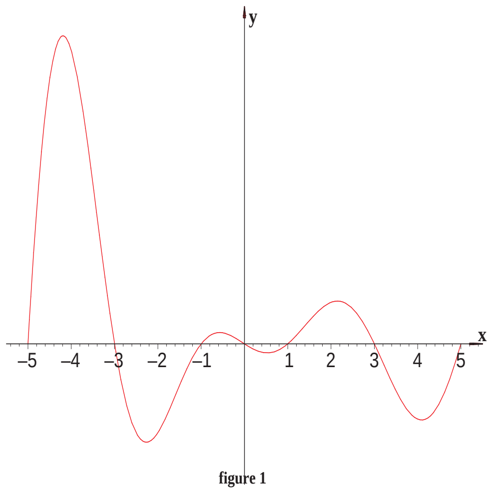


figure 1

T F (a) $g'(-4) = 0$.

T F (b) $g(-2) > 0$.

T F (c) $g''(2) > 0$.

T F (d) $g''(4) > 0$.

T F (e) $g'(-2) < 0$.

T F (f) $g'(0) < 0$.

32. For the function $g$ defined in Problem 31, find the points $x$ in the open interval $(-5, 5)$ at which $g$ has a local maximum, and the points at which $g$ has an absolute maximum over the closed interval $[-5, 5]$.

33. Let $a < c < b$ and let $g$ be differentiable on [a,b]. Which of the following is NOT necessarily true?

(a) $\displaystyle\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$, (b) There exists a $d$ in $[a, b]$ such that $g'(d) = \frac{g(b)-g(a)}{b-a}$, (c) $\displaystyle\int_a^b g(x)\,dx \geq 0$, (d) $\lim_{x\to c} g(x) = g(c)$, (e) If $k$ is a constant, then $\displaystyle\int_a^b k\,g(x)\,dx = k\int_a^b g(x)\,dx$.

34. If $f$ is an even and continuous function, then $\displaystyle\int_1^2 f(x)\,dx + \int_{-1}^{-2} f(x)\,dx =$

(a) $2\displaystyle\int_1^2 f(x)\,dx$, (b) $\displaystyle\int_{-1}^1 f(x)\,dx$, (c) 0, (d) $1/2$, (e) $\displaystyle\int_{-2}^2 f(x)\,dx$, (f) none of the above.

35. A publisher estimates that a book will sell at the rate of $r(t) = 16{,}000e^{-0.8t}$ books per year at the time $t$ years from now. Find the total number of books that will ever be sold (up to $t = \infty$).

36. Let $R$ be the region in the first quadrant enclosed by the $y$-axis and the graphs of $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq \pi/4$. (a) Set up the definite integral for the area of $R$ and evaluate it exactly. (b) Find the centroid of $(\overline{x}, \overline{y})$ of $R$. (c) Set up the integral for the volume of the solid generated when $R$ is revolved about the $x$-axis and evaluate it exactly. (d) Set up definite integrals to compute the perimeter of $R$. Do not compute the integrals.

37. For a function $f$ with continuous derivative $f'$ on the real line, the integral $\displaystyle\int x^3 f'(x^3)\,dx =$

(a) $x^3 f(x^3) - \displaystyle\int 3x^2 f(x^3)\,dx$, (b) $xf(x^3) - \displaystyle\int f(x^3)\,dx$, (c) $\displaystyle\int uf'(u)\,du$ with $u = x^3$, (d) $\dfrac{1}{3}xf(x^3) -$

$\dfrac{1}{3}\displaystyle\int f\left(x^3\right) dx$, (e) none of the above.

38. The amount of pollution in a lake $x$ years after the closing of a chemical plant is $P(x) = 100/x$ tons (for $x \geq 1$). Find the average amount of pollution between 1 and 10 years after the closing.

39. Consider the function $f(x) = 1 + x^2$ on the interval $[0, 2]$. Find a number $c$ in $[0, 2]$ so that the area of the rectangle with base on $[0, 2]$ and height $f(c)$ is equal to the area under the graph of $f$ in the given interval.

40. Compute the length of the curve given by $x = e^t \sin t$ and $y = e^t \cos t$, for $0 \leq t \leq \pi$.

41. A particle is moved along the $x$-axis by a force that measures $4x^2$ pounds at a point $x$ feet from the origin. Find the work done in moving the particle over a distance of 10 ft. from the origin.

42. A crane is lifting a 1500 lb transformer from the ground level to the third floor which is 30 feet above ground level. A 60 foot cable connects the transformer to the top of the crane. The cable weighs 5 lb per linear foot. How much work is done in lifting the transformer 30 feet above the ground?

43. The graph of a continuous function $f$ on the closed interval $[-5, 5]$ is shown in the following figure, where the arc is a semicircle. Let $h(x) = \displaystyle\int_{-1}^{x} f(t)\, dt$ for $-5 \leq x \leq 5$. (a) Compute $h(5)$ and $h(-4)$. (b) Compute $h'(-2)$ and $h'(3)$. (c) Find the set of points $x$ at which $h''$ is well-defined. (d) On what interval or intervals is the graph of $h$ concave upward? (e) Find the value(s) of $x$ at which $h$ has its absolute maximum and minimum on the closed interval $[-5, 5]$.

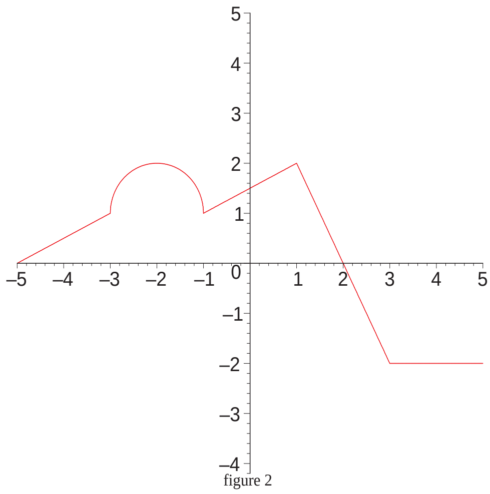


figure 2

44. It is observed that along a straight highway from city A to city B, a car passed city A at speed 30 mph (miles per hour) at 10:00 a.m. and passed city B at speed 50 mph (miles per hour) at 11:30 a.m. on the same day, where cities A and B are 100 miles apart.

T F (a) At a certain moment between 10:00 a.m. and 11:30 a.m., the car's speed has to be at least 70 mph.

T F (b) At a certain moment between 10:00 a.m. and 11:30 a.m., the car's speed has to be at least 66 mph.

T F (c) Between 10:00 a.m. and 11:30 a.m., the car's speed can never exceed 70 mph.

T F (d) At a certain moment between 10:00 a.m. and 11:30 a.m., the car's acceleration has to be at least 13 mi/h$^2$.

T F (e) Between 10:00 a.m. and 11:30 a.m., the car's acceleration can never be greater than 15 mi/h$^2$.

T F (f) Between 10:00 a.m. and 11:30 a.m., the car's acceleration can never be negative.

45. Find the volume of the solid obtained by revolving, about the line $x = 3$, the region $R$ in the first quadrant and bounded by the curves $y = 2x$ and $y = x^{1/5}$.

46. Find the volume of the solid $S$ that has the region $\left\{(x, y) : x^4/5 \le y \le 1\right\}$ in the $xy$-plane as its base and has all of its cross-sections perpendicular to the $y$-axis being squares.

47. Find the volume of the solid obtained by revolving, about the $y$-axis, the region

$$R = \left\{(x, y) : 1 \le x \le 2 \text{ and } 0 \le y \le e^{x^2}\right\}.$$

(Hint: Use the method of cylindrical shells.)

48. An aquarium 5 m long, 3 m wide, and 4 m deep is full of water. Find the work needed to pump half of the water out of the aquarium over its top. Note that the density of water is 1000 kg/m$^3$ and the gravitational acceleration is 9.8 m/s$^2$.

49. A swimming pool is 10 m wide and 15 m long, and its bottom is an inclined plane, the shallow end having a depth of 1 m and the deep end 3 m. If the pool is full of water, find the hydrostatic force on (a) the deep end, (b) one of the two (trapezoidal) sides, and (c) the bottom of the pool.

50. Let $y = \ln\left(2x + e^3\right)$. (i) Find the differential $dy$. (ii) Use $dy$ at $x = 0$ to estimate $\ln\left(e^3 + 0.02\right) - 3$ in terms of the constant $e$.

51. The edge of a cube is measured to be 10 cm with possible error in measurement of 0.1 cm. Use differential to estimate the maximum possible error in computing the surface area of the cube.

52. A man starts walking north at 4 ft/s from point $P$. Five seconds later a woman starts walking east at 3 ft/s from a point 20 ft due east of $P$. At what rate are these two persons moving apart 10 seconds after the woman starts walking?

53. Find $f(x)$ for $x > 0$, if $f(1) = 3$, $f(2) = 2$, and $f''(x) = -12x + x^{-2}$.

54. With what constant negative acceleration $a_0$ (ft/s$^2$) by brakes can a car be brought to a full stop from a speed of 60 mi/h within exactly a distance of 50 feet? (1 mi. = 5280 ft.)

55. Find the volume of the solid $S$ with a flat base which is the region $R$ bounded by $y = x^2$ and $y = 4x$ on the $xy$-plane and with its intersection with any plane $x = c$, $c \in \mathbb{R}$, being either an equilateral triangle or an empty set.

56. If $\int_0^3 f(x)\,dx = 1$, $\int_6^8 f(x)\,dx = -3$, and $\int_0^8 f(x)\,dx = 4$, find (i) $\int_3^6 f(x)\,dx$ and (ii) $\int_0^8 (2f(x) - 3\sin(x))\,dx$.

57. Find the area of the region $R$ bounded by the curves $y = x^2 - 4x$ and $y = 2x - x^2$.

58. A log 10 meters long is cut at 1-meter intervals and the diameters, in meters, of its (circular) cross sections at these 9 cuts from one end to the other are 0.25, 0.28, 0.26, 0.27, 0.28, 0.3, 0.29, 0.28, and 0.29. Use Simpson's Rule to estimate the volume of the log.

59. Compute (i) $\displaystyle\int \frac{2x}{3x^2+1}dx$, (ii) $\displaystyle\int_0^3 x\sqrt{x+1}dx$, (iii) $\displaystyle\int \sin^4(x)\cos^3(x)\,dx$, (iv) $\displaystyle\int \sin^2(3x)\,dx$, (v) $\displaystyle\int_0^3 \frac{1}{(x-1)^4}dx$, (vi) $\displaystyle\int \frac{x-2}{x^2-9}dx$, and (vii) $\displaystyle\int x\left(\sqrt{x^2+1}\right)\ln\left(x^2+1\right)dx$.

60. Which of the following improper integrals is convergent?

(a) $\displaystyle\int_1^\infty x^{-3/2}\sin^2(x)\,dx$, (b) $\displaystyle\int_1^\infty xe^{-x}\,dx$, (c) $\displaystyle\int_1^\infty \frac{1}{x(1+x^2)}dx$, (d) $\int_0^1 x^{-1/2}(1-x)\,dx$, (e) all of the above, (f) none of the above.