

Math 141 Homework #12  
Due Tuesday, 11/13/07

**Bonus Problem:** The extremely important proof technique of *mathematical induction* is often used to show that some fact is true for every positive integer. For example, the sums

$$S(n, p) = \sum_{i=1}^n i^p$$

that come up in integration have closed-form formulas like

$$(1) \quad S(n, 1) = \sum_{i=1}^n i = \frac{n(n+1)}{2},$$

$$(2) \quad S(n, 2) = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$(3) \quad S(n, 3) = \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

To prove that these formulas work for every positive integer  $n$ , you can use induction. Induction is vital in many other areas of mathematics, including sequences and series (which you'll see in Math 122/142) and identities involving binomial coefficients (think back to the first few days of class).

The technique of induction can be a bit confusing at first, but with a little practice, you can get used to how it works, and it is very much worth learning. A brief summary appears in the box on p.87 of the textbook, although it's not very enlightening by itself; you really have to work through a couple of examples to get the hang of how the technique works. The Wikipedia article on induction at

[http://en.wikipedia.org/wiki/Mathematical\\_induction](http://en.wikipedia.org/wiki/Mathematical_induction)

has a more detailed description of induction that is readable and accurate, including a proof of the formula (1) (which is the "standard" example of induction). The formula (2) is proved inductively in Appendix F on p. A47. A slightly different example of induction appears on p. 90.

On the other hand, it is possible to prove formulas like (1), (2) and (3) without induction, as in Examples 4 and 5 on pp. A46–A47.

Read all this material. Once you have done so, mimic the method of Example 5 to find a formula for  $S(n, 4)$ , and check that it works for several values of  $n$  (say  $1 \leq n \leq 5$ ). (If you want, you can check your formula for  $S(n, 4)$  against the Wikipedia article on Faulhaber's formula; see below.)

Next, give another proof of your formula by induction.

Finally, find a recursive formula for  $S(n+1, p)$  in terms of  $S(n, p)$ , again by mimicking the method of Example 5. (Hint: In order to make the method work for all  $n$ , you will need binomial coefficients.)

For the curious, there is a general formula for  $S(n, 4)$  called Faulhaber's formula; see

[http://en.wikipedia.org/wiki/Faulhaber's\\_formula](http://en.wikipedia.org/wiki/Faulhaber's_formula).

However, the formula involves things called Bernoulli numbers, which are not easy to write in closed form; indeed, to give a general formula for them, you need mathematical induction again!