The Traveling Salesman Problem

Definition: A **complete graph** K_N is a graph with N vertices and an edge between every two vertices.

Definition: A **weighted graph** is a graph in which each edge is assigned a *weight* (representing the time, distance, or cost of traversing that edge).

Definition: A **Hamilton circuit** is a circuit that uses every vertex of a graph once.

Definition: The **Traveling Salesman Problem (TSP)** is the problem of finding a **minimum-weight Hamilton circuit** in K_N.

The Traveling Salesman Problem

Example: Willy decides to visit every Australian city important enough to be listed on this Wikipedia page.

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What route minimizes the total distance he has to travel?

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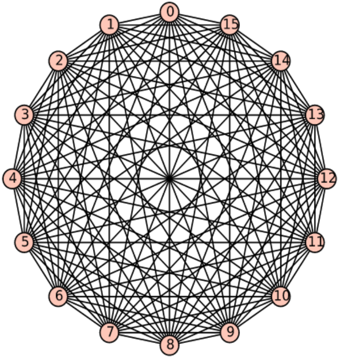
To avoid rental-car fees, he must finish the tour in the same city he starts it in.

What route minimizes the total distance he has to travel?

I.e., in this weighted K_{16}, which Hamilton circuit has the smallest total weight?

The Traveling Salesman Problem

Since K_{16} is difficult to draw (imagine labeling all these edges!), we will just use the distance table.



The Traveling Salesman Problem

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Australian Distance Table

Adelaide	http://en.wikipedia.org/wiki/States_and_territories_of_Australia#Distance_table														
2673	Albany														
1533	3588	Alice Springs													
1578	3633	443	Uluru												
2045	4349	3038	3254	Brisbane											
2483	1943	2483	1223	3317	Broome										
3352	5656	2457	2900	1716	2496	Cairns									
1196	3846	3706	2751	1261	3275	2568	Canberra								
3022	4614	1489	1932	3463	1803	2882	4195	Darwin							
1001	3674	2534	2579	1944	3636	3251	918	4023	Hobart						
3219	3787	1686	2129	3660	1045	3079	4392	827	4220	Kununurra					
2783	5087	2505	2948	976	2840	740	1999	2930	2682	3127	Mackay				
731	3404	2264	2309	1674	3124	2981	648	3753	609	3950	2412	Melbourne			
2742	5106	1209	1652	1829	1834	1248	2561	1634	3075	1831	1296	2805	Mount Isa		
2781	409	3696	3741	4457	2389	5764	3954	4205	3782	3378	5195	3512	4905	Perth	
1412	3970	3830	2875	1001	3373	2495	286	4034	1142	4516	1926	872	2400	4078	Sydney

The Brute-Force Algorithm

Willy could solve the problem by **brute force**:

1. Make a list of all possible Hamilton circuits.
2. Calculate the weight of each Hamilton circuit by adding up the weights of its edges.
3. Pick the Hamilton circuit with the smallest total weight.

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BIG PROBLEM! There are 16 vertices, so there are $15! = 1,307,674,368,000$ Hamilton circuits that each need to be checked. That's a lot of Hamilton circuits.

Solving the TSP Without Brute Force

Idea: At each stage in your tour, choose the closest vertex that you have not visited yet.

This is called the **Nearest-Neighbor Algorithm (NNA).**

This spreadsheet shows what happens when Willy uses the NNA to construct a Hamilton circuit (with Sydney as the reference vertex).

The Nearest-Neighbor Algorithm

The result: The Nearest-Neighbor algorithm, using Sydney as the reference vertex, yields the Hamilton circuit

SY $\rightarrow$ CN $\rightarrow$ ML $\rightarrow$ HO $\rightarrow$ AD $\rightarrow$ AS $\rightarrow$ UL $\rightarrow$ BM $\rightarrow$ KU $\rightarrow$ DA $\rightarrow$ MI $\rightarrow$ CS $\rightarrow$ MK $\rightarrow$ BR $\rightarrow$ AL $\rightarrow$ PE $\rightarrow$ SY

whose total weight is **21,049 km**.

A randomly chosen Hamilton circuit would have averaged **40,680 km**, so this is pretty good.

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But can Willy do better?

The Repetitive Nearest-Neighbor Algorithm

Observation: Willy can use **any city** as the reference vertex!

That is, Willy can execute the Nearest-Neighbor Algorithm sixteen times, using each city once as the reference vertex.

Then, he can pick the Hamilton circuit with the lowest total weight of these sixteen.

This is called the **Repetitive Nearest-Neighbor Algorithm (RNNA)**.

The Repetitive Nearest-Neighbor Algorithm

Ref. vertex	Hamilton circuit	Weight	
AD	AD,ML,HO,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,PE,AL,AD	18543	
AL	AL,PE,BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BR,MK,CS,MI,AL	19795	
AS	AS,UL,BM,KU,DA,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,AS	18459	
BR	BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,AL,PE,BR	22113	
BM	BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BR,MK,CS,MI,PE,AL,BM	19148	
CS	CS,MK,BR,SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,PE,AL,CS	22936	
CN	CN,SY,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BR,AL,PE,CN	21149	
DA	DA,KU,BM,UL,AS,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,DA	18543	
HO	HO,ML,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,HO	20141	
KU	KU,DA,AS,UL,BM,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,KU	18785	
MK	MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,BR,AL,PE,MK	23255	
ML	ML,HO,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,ML	20141	
MI	MI,AS,UL,BM,KU,DA,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,MI	20877	
PE	PE,AL,BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BR,MK,CS,MI,PE	19148	
SY	SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BR,AL,PE,SY	21049	(NNA)
UL	UL,AS,MI,CS,MK,BR,SY,CN,ML,HO,AD,BM,KU,DA,PE,AL,UL	20763	

The Repetitive Nearest-Neighbor Algorithm

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AD	AD,ML,HO,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,PE,AL,AD	18543	
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AS	AS,UL,BM,KU,DA,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,AS	18459	**(best)**
BR	BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,AL,PE,BR	22113	
BM	BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BR,MK,CS,MI,PE,AL,BM	19148	
CS	CS,MK,BR,SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,PE,AL,CS	22936	
CN	CN,SY,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BR,AL,PE,CN	21149	
DA	DA,KU,BM,UL,AS,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,DA	18543	
HO	HO,ML,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,HO	20141	
KU	KU,DA,AS,UL,BM,MI,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,KU	18785	
MK	MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,BR,AL,PE,MK	23255	
ML	ML,HO,CN,SY,BR,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,ML	20141	
MI	MI,AS,UL,BM,KU,DA,CS,MK,BR,SY,CN,ML,HO,AD,AL,PE,MI	20877	
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SY	SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BR,AL,PE,SY	21049	(NNA)
UL	UL,AS,MI,CS,MK,BR,SY,CN,ML,HO,AD,BM,KU,DA,PE,AL,UL	20763	

The Repetitive Nearest-Neighbor Algorithm

Apparently, using Alice Springs (AS) as the reference vertex yields the best Hamilton circuit so far, namely

AS $\rightarrow$ UL $\rightarrow$ BM $\rightarrow$ KU $\rightarrow$ DA $\rightarrow$ MI $\rightarrow$ CS $\rightarrow$ MK $\rightarrow$ BR $\rightarrow$ SY $\rightarrow$ CN $\rightarrow$ ML $\rightarrow$ HO $\rightarrow$ AD $\rightarrow$ AL $\rightarrow$ PE $\rightarrow$ AS

The Repetitive Nearest-Neighbor Algorithm

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AS $\rightarrow$ UL $\rightarrow$ BM $\rightarrow$ KU $\rightarrow$ DA $\rightarrow$ MI $\rightarrow$ CS $\rightarrow$ MK $\rightarrow$ BR
$\rightarrow$ SY $\rightarrow$ CN $\rightarrow$ ML $\rightarrow$ HO $\rightarrow$ AD $\rightarrow$ AL $\rightarrow$ PE $\rightarrow$ AS

Remember: Willy can still start anywhere he wants!
For instance,

SY $\rightarrow$ CN $\rightarrow$ ML $\rightarrow$ HO $\rightarrow$ AD $\rightarrow$ AL $\rightarrow$ PE $\rightarrow$ AS
$\rightarrow$ UL $\rightarrow$ BM $\rightarrow$ KU $\rightarrow$ DA $\rightarrow$ MI $\rightarrow$ CS $\rightarrow$ MK $\rightarrow$ BR $\rightarrow$ SY

represents the same Hamilton circuit.

The Repetitive Nearest-Neighbor Algorithm

Randomly chosen Hamilton circuit:	40,680 km
Hamilton circuit using NNA/Sydney:	21,049 km
Hamilton circuit using RNNA:	**18,459 km**

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- In general, there's no way of knowing in advance which reference vertex will yield the best result.

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- This algorithm is still **efficient**, but ...

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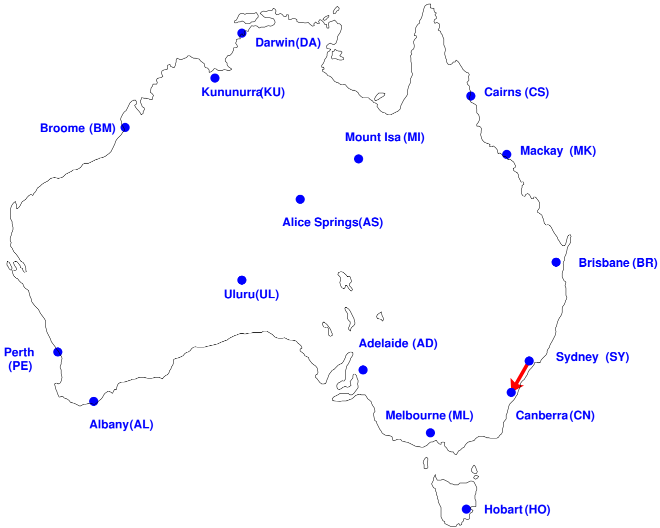
Hamilton circuit using RNNA:	**18,459 km**

- In general, there's no way of knowing in advance which reference vertex will yield the best result.
- This algorithm is still **efficient**, but ...
- Is it **optimal?** That is, Can Willy do even better?

The Repetitive Nearest-Neighbor Algorithm

If we look at the map (warning: not quite to scale!) it becomes clear that the RNNA has not produced an optimal Hamilton circuit.



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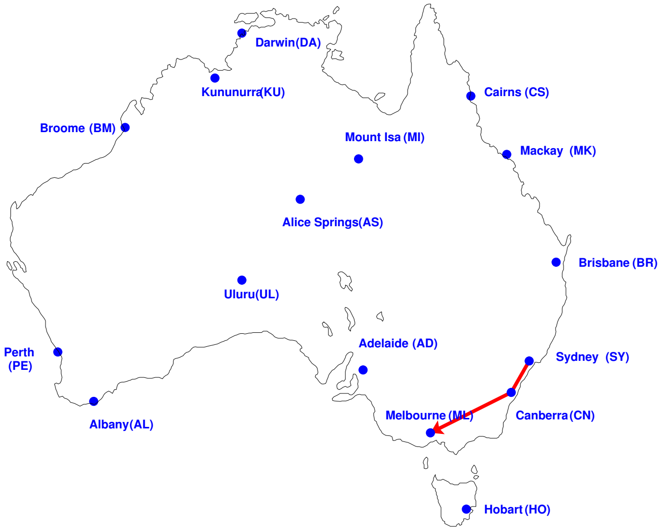

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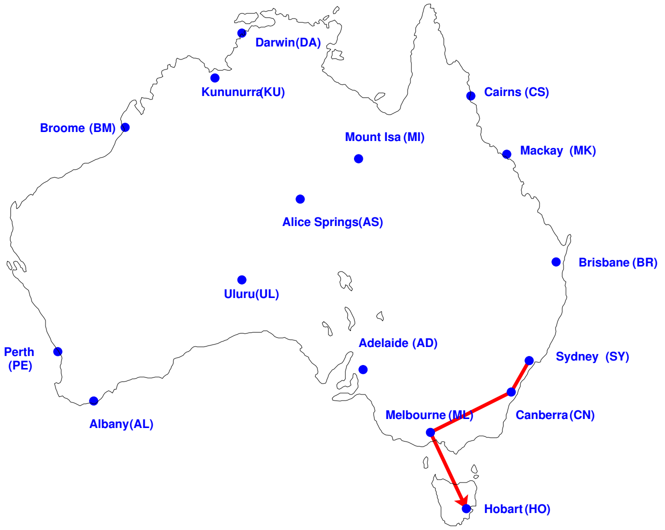
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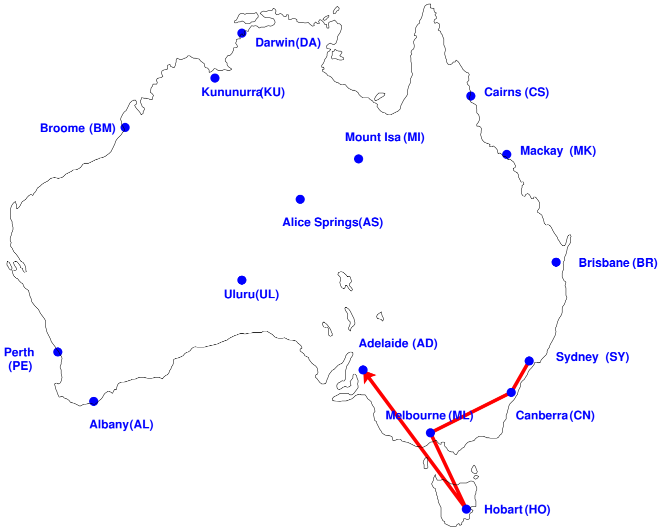
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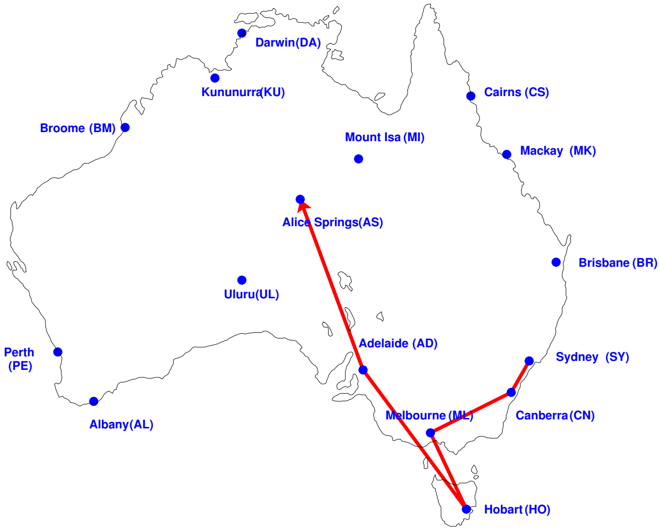
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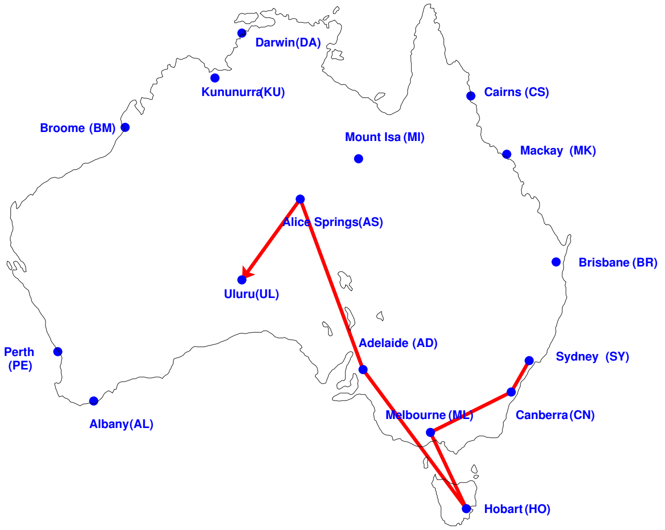
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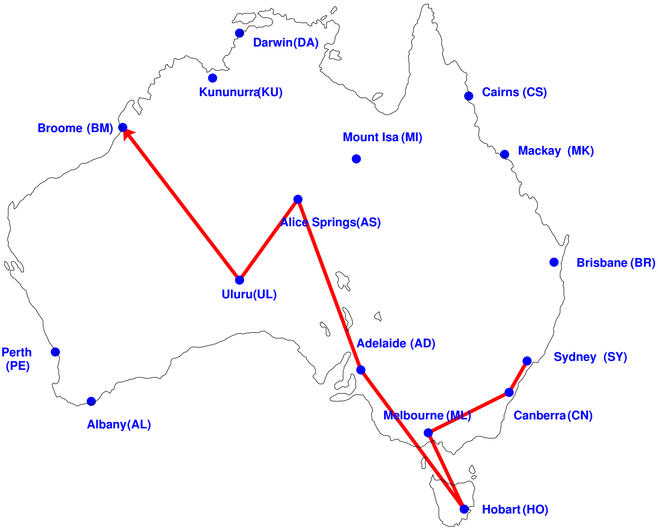
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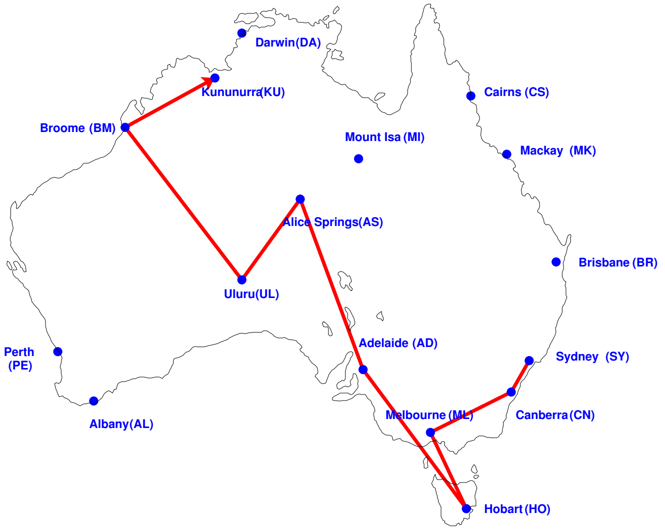
Oops.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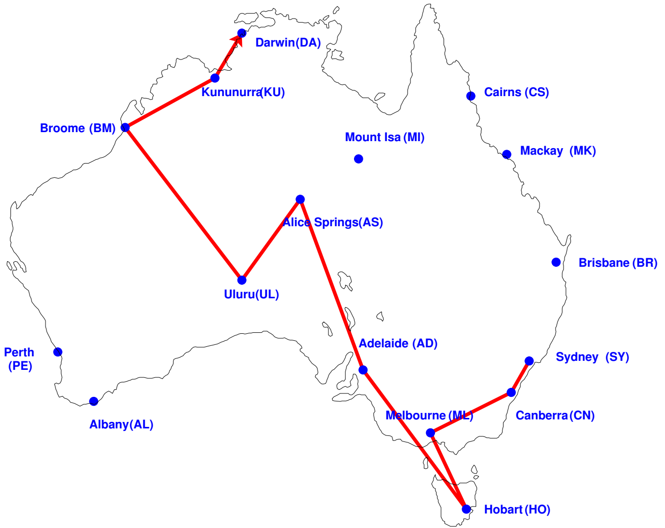

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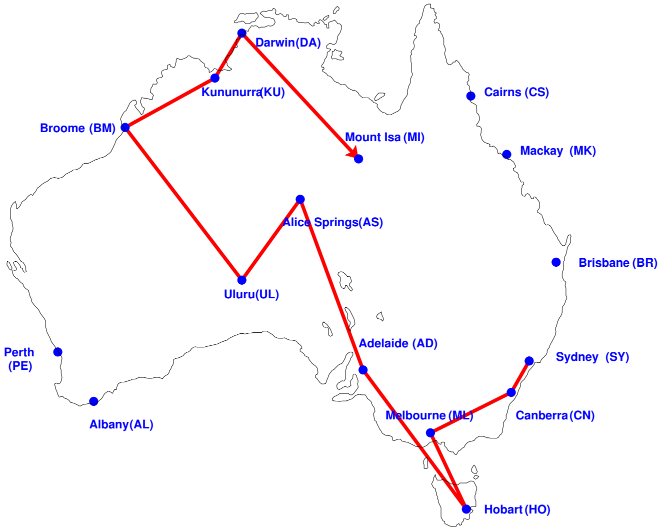
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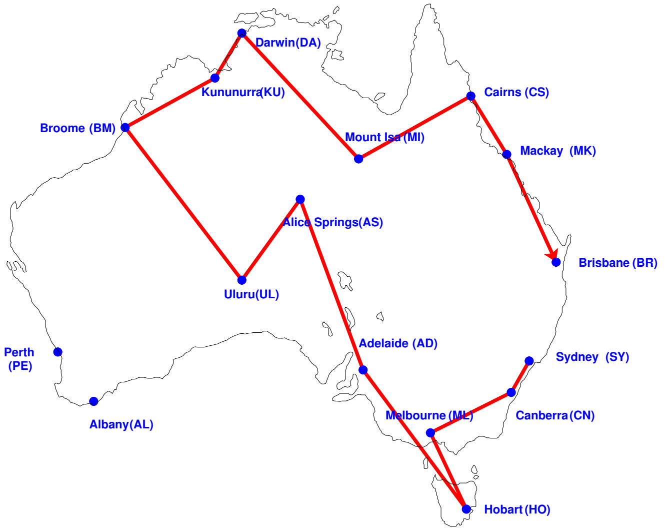

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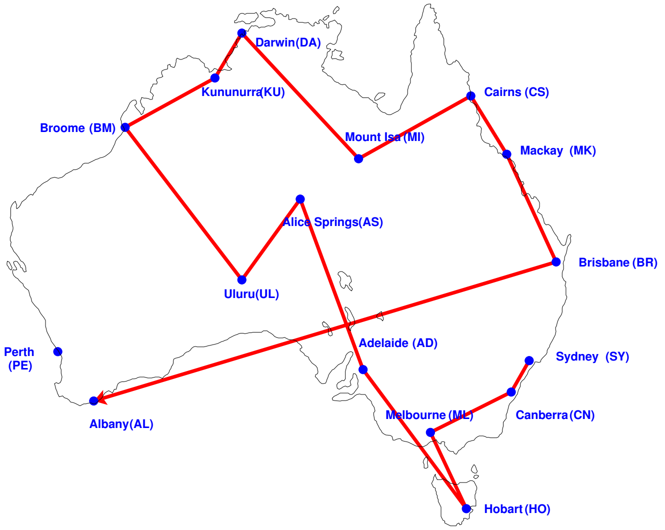
Now the algorithm is stuck — some very expensive edges are required to complete the Hamilton circuit.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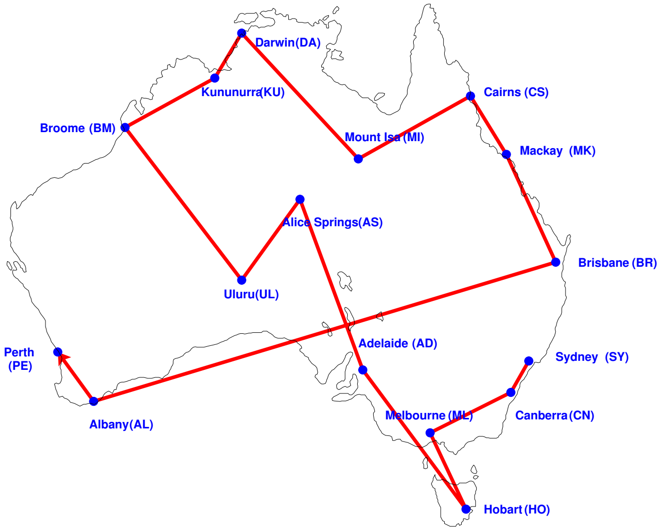
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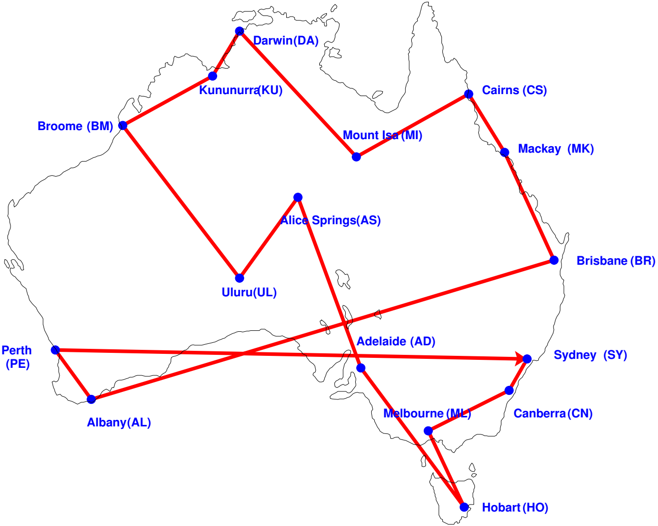
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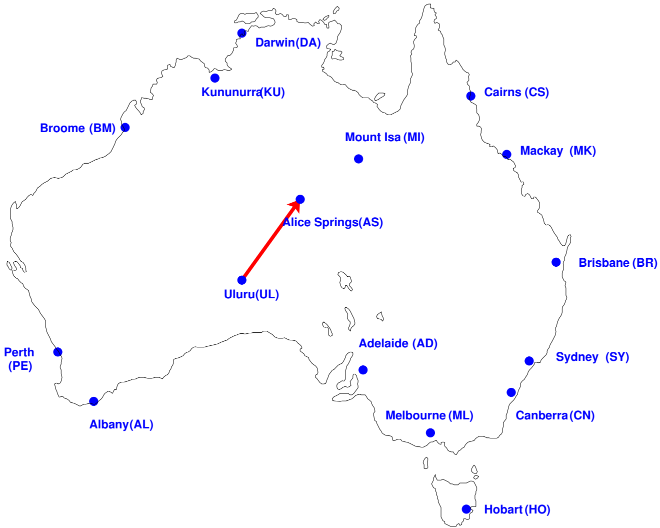
The Repetitive Nearest-Neighbor Algorithm

Starting from Uluru would have created a different problem.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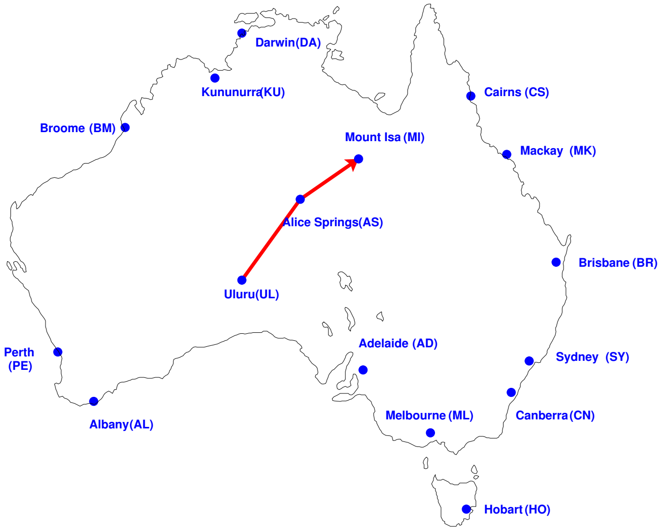
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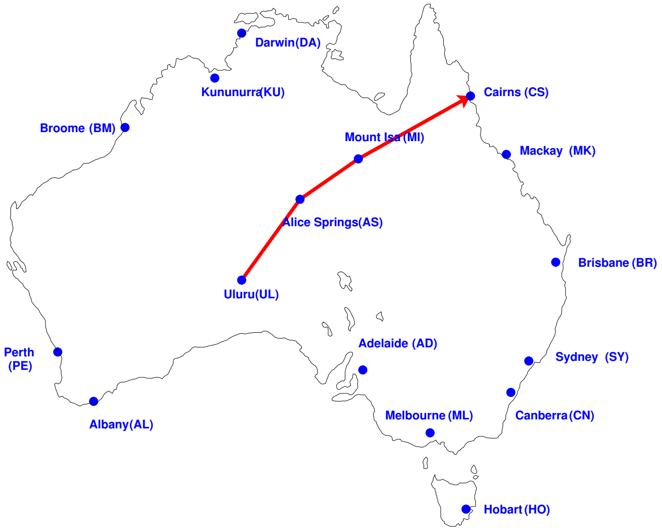
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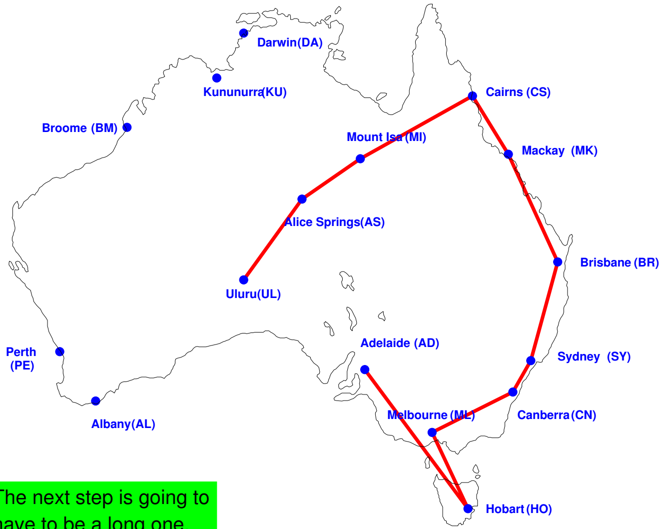
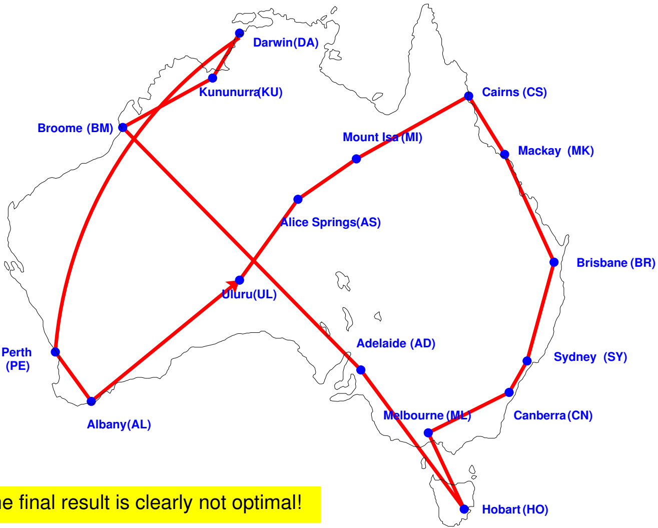


The next step is going to have to be a long one.

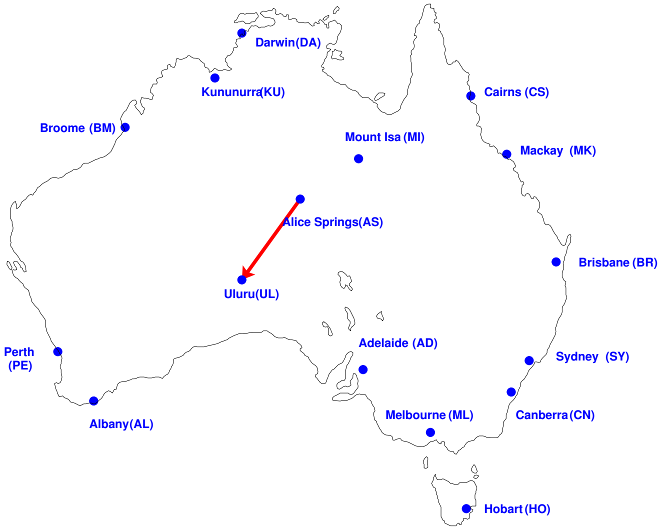
Uh–oh.
We can see that going to Albany next would be best – but Broome is closer. That's what the algorithm picks.



The final result is clearly not optimal!

The Repetitive Nearest-Neighbor Algorithm

Starting from Alice Springs would have created a different problem (but a less harmful one).



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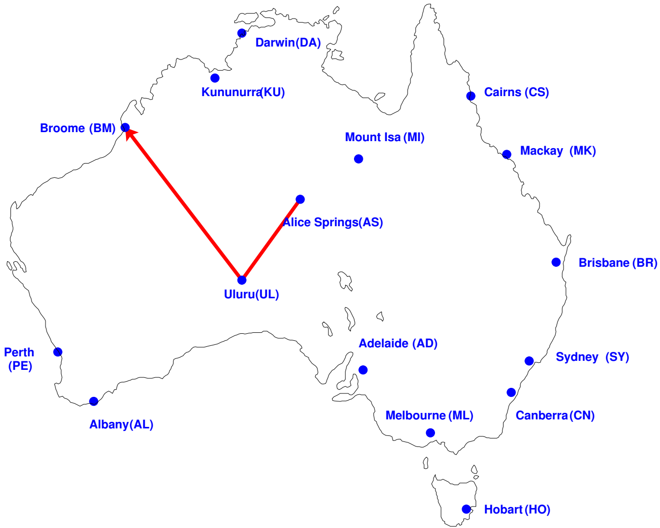
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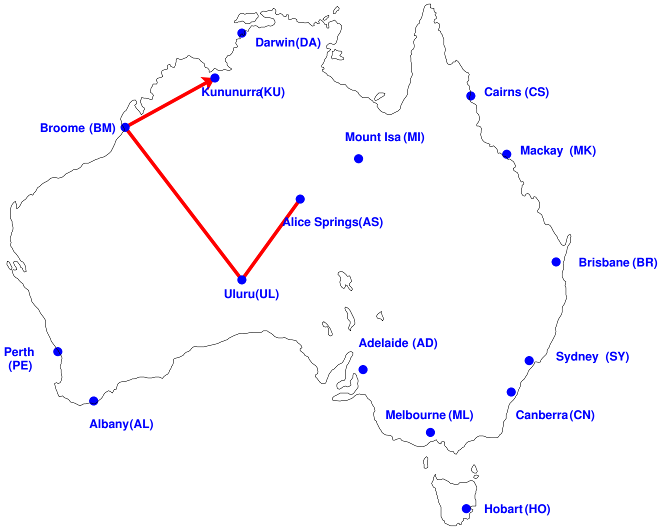
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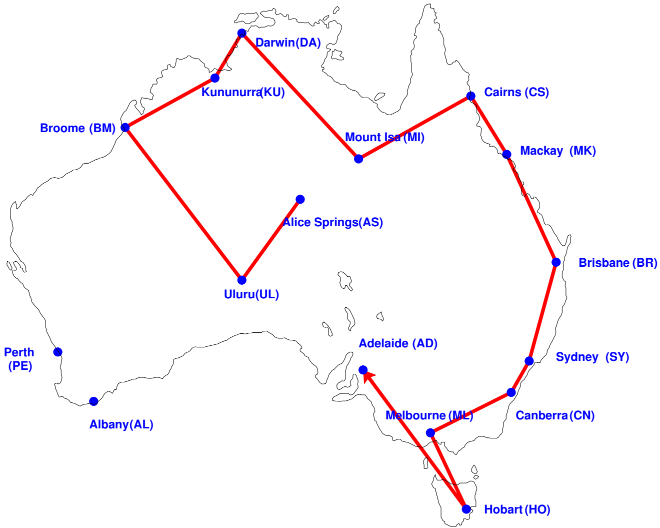
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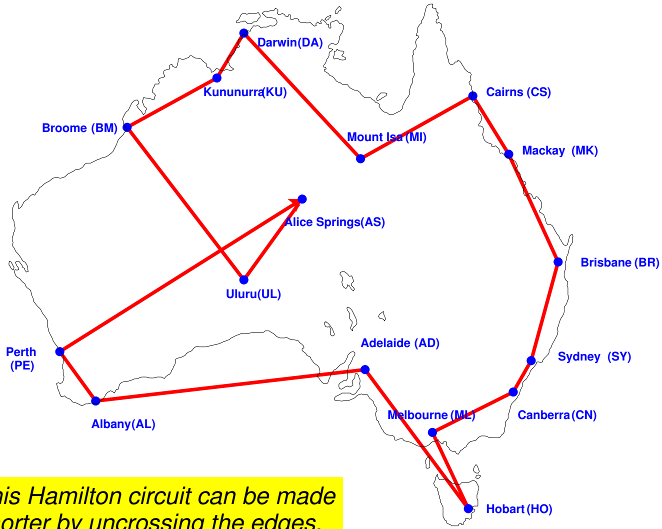
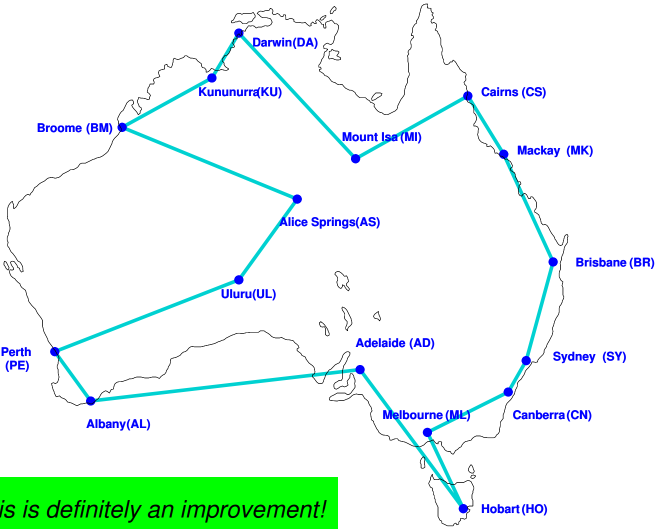


This Hamilton circuit can be made shorter by uncrossing the edges.



This is definitely an improvement!

The Repetitive Nearest-Neighbor Algorithm

It is often easy for a human to look at the Hamilton circuit produced by an algorithm and find ways to improve it.

The hard part is to do it **systematically**.

It is also hard to be sure that even after making improvements (such as "uncrossings"), the resulting Hamilton circuit is optimal. (In fact, this one probably isn't.)

What about another way of trying to construct an efficient Hamilton circuit?