The Mathematics of Touring (Chapter 6)

- In Chapter 5, we studied **Euler paths** and **Euler circuits**: paths and circuits that use every **edge** of a graph.
- In Chapter 6, we'll look at circuits that use every **vertex** of a graph exactly once. These are called **Hamilton circuits.**
- Instead of asking, "Does a graph have a Hamilton circuit?", the interesting question is often, **"Out of all the possible Hamilton circuits, which one is the most efficient?"**

The Traveling Salesman Problem (TSP)

Willy, a traveling salesman, has to visit each of several cities (say, the 48 state capitals of the continental United States)

He would like his trip to cover as little distance as possible.

In what order should Willy visit the 48 cities?

This problem is the **Traveling Salesman Problem**, or **TSP.**

Note that there are many possible routes — to be exact, there are $47! \approx 2.6 \times 10^{59}$ of them. The problem is to find the **shortest** of these routes.

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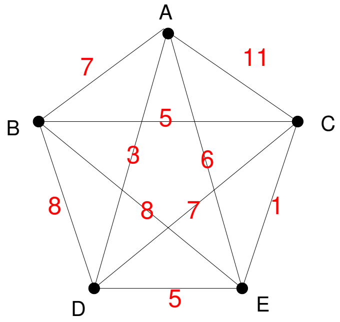
- You are taking your four-year-old trick-or-treating and needs to visit each of eight friends and relatives

The TSP As A Graph Problem

Suppose we have a graph in which every edge has a **weight** (representing its cost, time, or distance).



The TSP is then to find a path or a circuit that

- visits every vertex; and
- has total weight as low as possible.

Hamilton Paths and Hamilton Circuits

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A **Hamilton circuit** is a circuit that uses **every vertex** of a graph **exactly once.**

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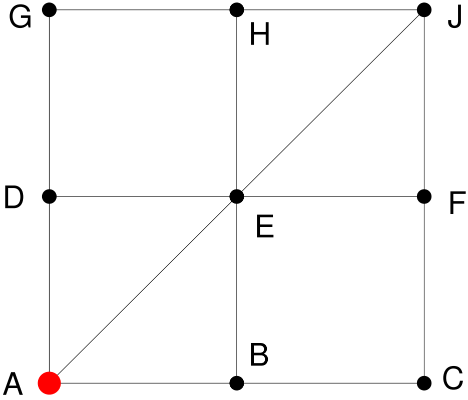
A **Hamilton circuit** is a circuit that uses **every vertex** of a graph **exactly once.**

- By contrast, an Euler path/circuit is a path/circuit that uses every *edge* exactly once. (Mnemonic: **E**uler = **E**dge)

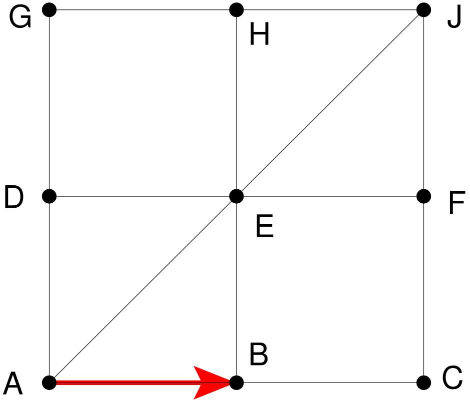
- "Path": starting and ending vertices are different.
- "circuit" starting and ending vertices are the same.

Hamilton Paths and Hamilton Circuits



Start at A
("reference vertex")

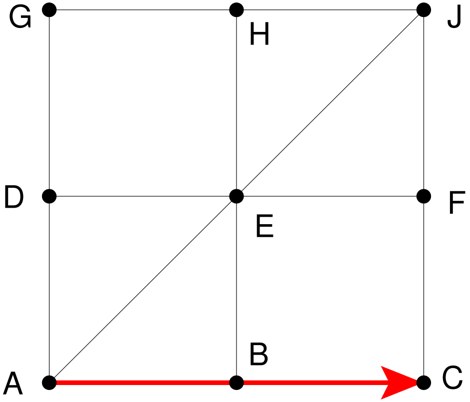
Hamilton Paths and Hamilton Circuits



Path so far:

AB

Hamilton Paths and Hamilton Circuits



Path so far:

ABC

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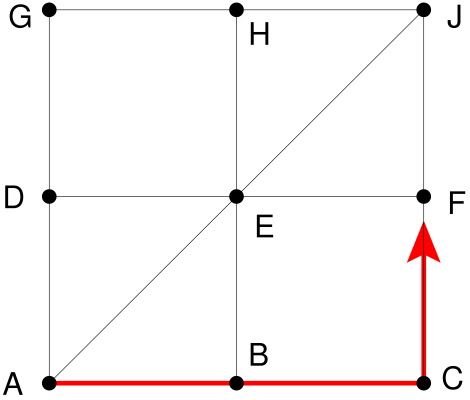
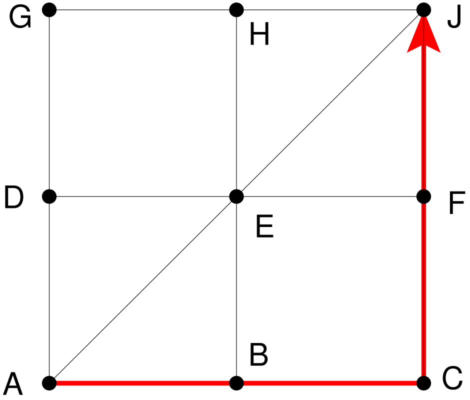


Path so far:

ABCF

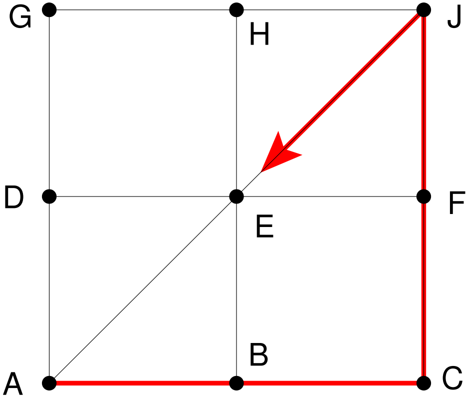
Hamilton Paths and Hamilton Circuits



Path so far:

ABCFJ

Hamilton Paths and Hamilton Circuits



Path so far:

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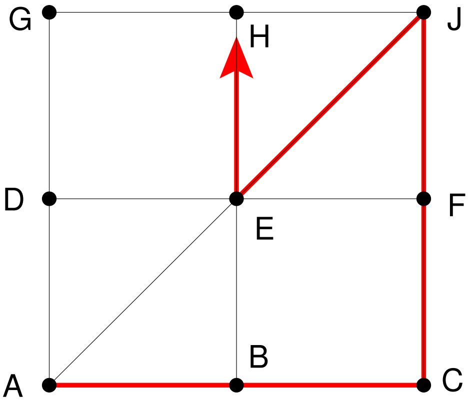

Path so far:

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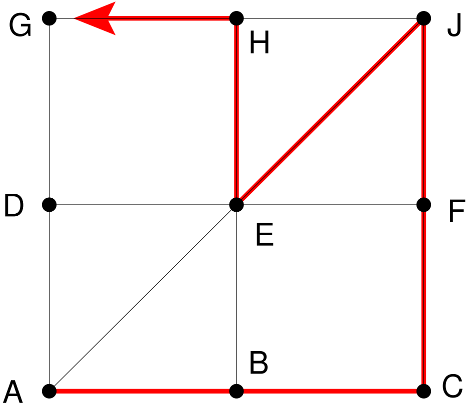

Path so far:

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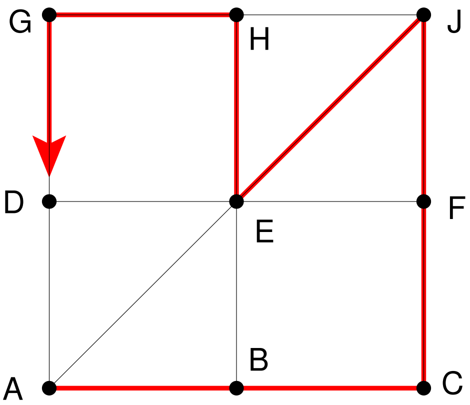
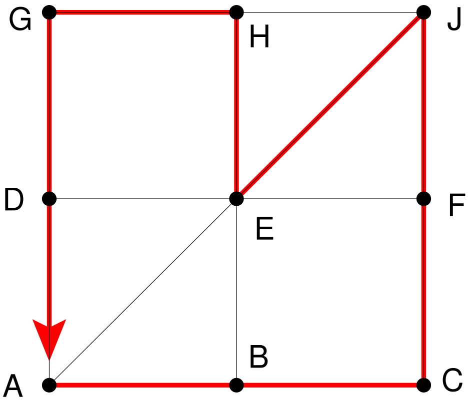


Path so far:

ABCFJEHGD

Hamilton Paths and Hamilton Circuits



Path so far:

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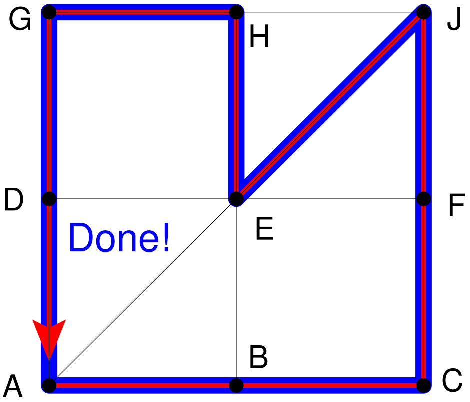
Hamilton Paths and Hamilton Circuits



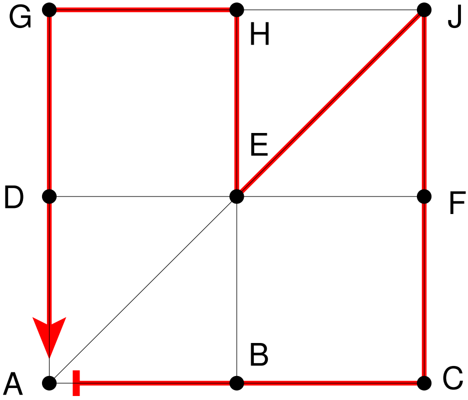
Complete circuit:

ABCFJEHGDA

Hamilton Paths and Hamilton Circuits

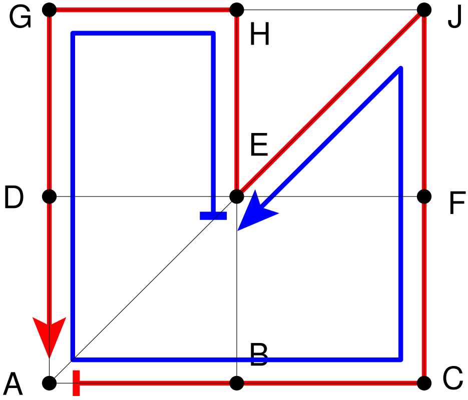
Changing the starting vertex (or "reference vertex") does not change the Hamilton circuit, because the same edges are traversed in the same directions.

Hamilton Paths and Hamilton Circuits



Reference vertex A
Hamilton circuit:
ABCFJEHGDA

Hamilton Paths and Hamilton Circuits



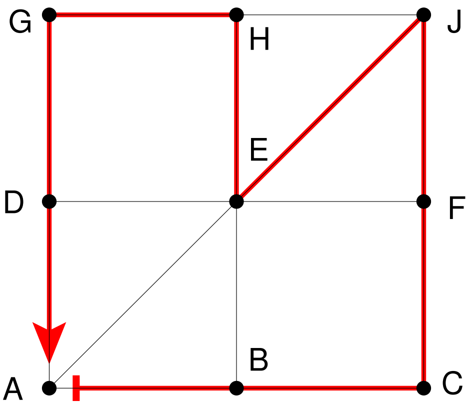
Reference vertex A
Hamilton circuit:
ABCFJEHGDA

Reference vertex E
Hamilton circuit:
EHGDABCFJE

Hamilton Paths and Hamilton Circuits

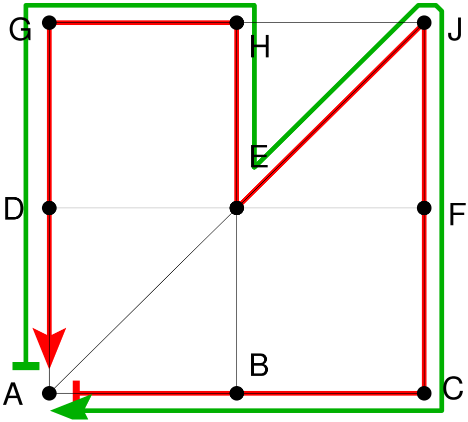
We can also make a Hamilton circuit into its “mirror image” by reversing direction. The mirror image uses the same edges, but **backwards**, so it is not considered the same as the original Hamilton circuit.

Hamilton Paths and Hamilton Circuits



Reference vertex A
Hamilton circuit:
ABCFJEHGDA

Hamilton Paths and Hamilton Circuits



Reference vertex A
Hamilton circuit:
ABCFJEHGDA

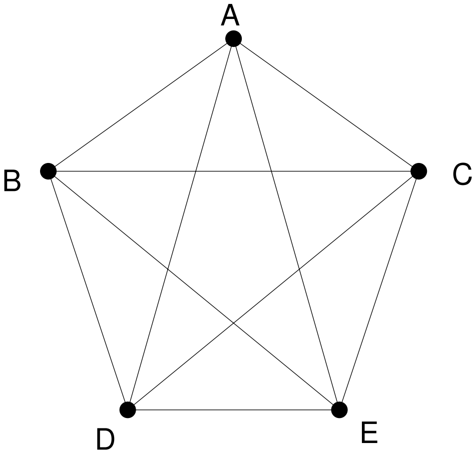
Reverse direction
Hamilton circuit:
ADGHEJFCBA

Hamilton vs. Euler

Can a graph have both a Hamilton circuit and an Euler circuit?

Hamilton vs. Euler

Can a graph have both a Hamilton circuit and an Euler circuit? ★



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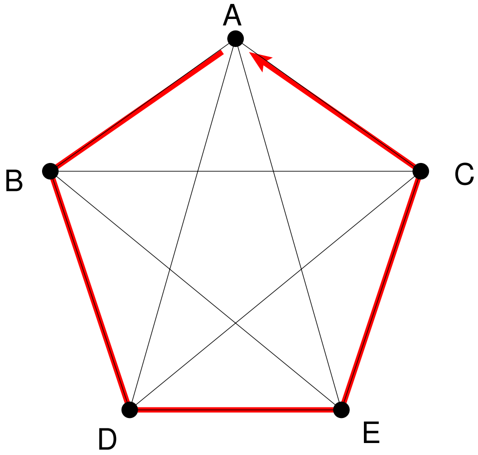

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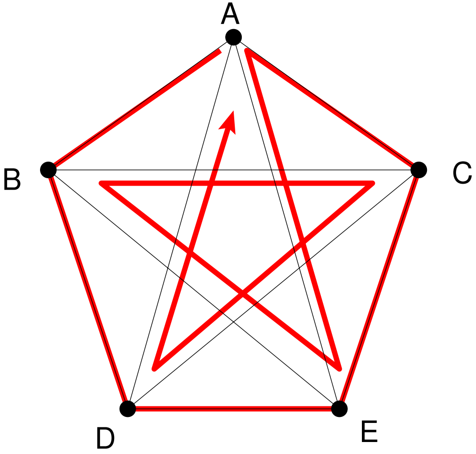

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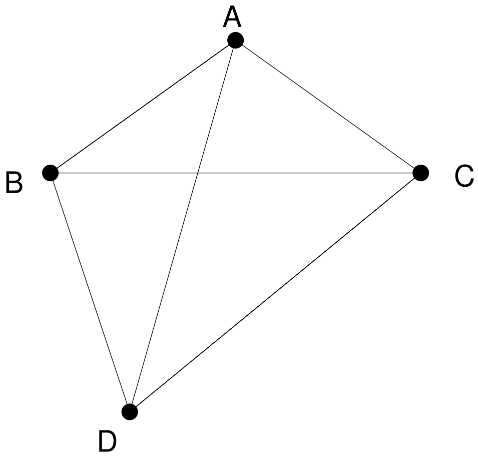

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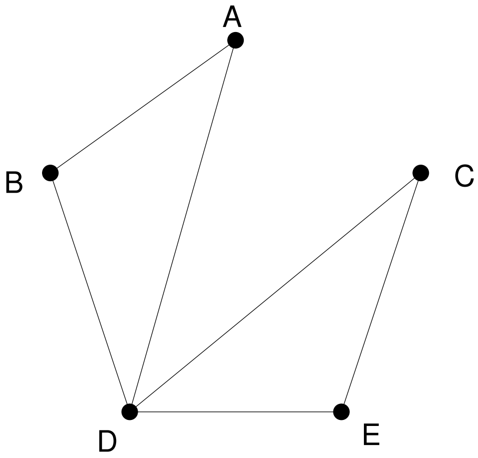

Hamilton vs. Euler

Can a graph have neither a Hamilton circuit nor an Euler circuit? (Let's just consider connected graphs.)

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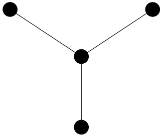

This graph has no circuits at all!

Hamilton vs. Euler

Conclusion: Whether a graph does or does not have a Hamilton circuit **tells you nothing** about whether it has an Euler circuit, and vice versa.

The same is true for Hamilton/Euler **paths** (rather than circuits).

Which Graphs Have Hamilton Circuits?

We know how to determine whether a graph has an Euler path or circuit: count the odd vertices.

On the other hand, **there is no simple way to tell whether or not a given graph has a Hamilton path or circuit**.

Finding the Shortest Hamilton Circuit

Rather than asking whether a particular graph has a Hamilton circuit, we will be looking at graphs with **lots** of Hamilton circuits, and trying to find the **shortest** one.

For example, Willy the traveling salesman has the option to drive from any state capital to any other, so the relevant graph has lots of edges — it is a **complete graph**.