Agenda

1. Counting formulas for complete graphs
2. Working with factorials
3. Counting spanning trees in networks
4. Kruskal's algorithm

Note: Graph coloring (the "Mini-Excursion") will **not** be on Monday's test.

1. Formulas for Complete Graphs

In the complete graph K_N:

- Number of edges in the whole graph: $N(N-1)/2$
- Number of edges in a spanning tree: $N-1$
- Number of Hamilton circuits: $(N-1)!$
- Number of spanning trees: N^{N-2} (Cayley's formula)

1. Formulas for Complete Graphs

Number of edges in $K_N = N(N-1)/2$.

- This is the same as the formula for the number of pairwise comparisons among N candidates.

1. Formulas for Complete Graphs

Number of edges in a spanning tree of $K_N = N - 1$

- This is because in **any** connected graph with N vertices, a spanning tree must have exactly $N - 1$ edges.

1. Formulas for Complete Graphs

Number of Hamilton circuits in $K_N = (N-1)!$

- After choosing the reference vertex, there are $N-1$ possibilities for the second vertex, $N-2$ possibilities for the third vertex, $\ldots$, 2 possibilities for the second-to-last vertex, 1 possibility for the last vertex.
- Multiplying all these numbers together gives $(N-1)(N-2)\cdots(2)(1) = (N-1)!$

1. Formulas for Complete Graphs

Number of spanning trees in $K_N = N^{N-2}$.

- We worked several examples out in class:

  K_1: 1 spanning tree ($= 1^{-1}$)

  K_2: 1 spanning tree ($= 2^0$)

  K_3: 3 spanning trees ($= 3^1$)

  K_4: 16 spanning trees ($= 4^2$)

  K_5: 125 spanning trees ($= 5^3$)

  …

  K_N: N^{N-2} spanning trees

2. Factorials

$$
\begin{aligned}
1! &= 1 = \mathbf{1} \\
2! &= 1 \times 2 = \mathbf{2} \\
3! &= 1 \times 2 \times 3 = \mathbf{6} \\
4! &= 1 \times 2 \times 3 \times 4 = \mathbf{24} \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 = \mathbf{120} \\
&\vdots \\
12! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \\
&= \mathbf{479001600} \\
13! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \\
&= \mathbf{6227020800}
\end{aligned}
$$

2. Factorials

$70! =$
$1197857166996989179607278372168909873645893814254642585755536286462800958278984531968000000000000000$

This is too many digits for most calculators.

Even if your calculator could count this high, it would probably display it as something like

$$1.1978571669\text{E}100.$$

2. Factorials

Problem #1: Simplify $4!/3!$ without using a calculator.

Problem #2: Simplify $13!/12!$ without using a calculator.

2. Factorials

Problem #1: Simplify $4!/3!$ without using a calculator.

$$\begin{aligned} 3! &= 1 \times 2 \times 3 = 6 \\ 4! &= 1 \times 2 \times 3 \times 4 = 24 \\ 4!/3! &= 24/6 = 4 \end{aligned}$$

But also,

$$\frac{4!}{3!} = \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3} = 4.$$

2. Factorials

Similarly,

$$\frac{13!}{12!} = 6227020800/479001600 \quad \text{(yuck)}$$

$$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12}$$

$$= 13.$$

In general,

$$\boxed{\frac{\mathbf{N!}}{\mathbf{(N-1)!}} = \mathbf{N.}}$$

2. Factorials

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$$= 13.$$

In general,

$$\boxed{\frac{\mathbf{N!}}{\mathbf{(N-1)!}} = \mathbf{N.}}$$

2. Factorials

Sample problems:

$$\frac{6!}{5!} = \square$$

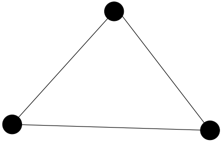
$$\frac{100!}{98!} = \square$$

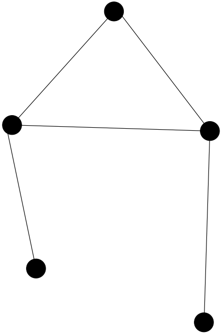
$$\frac{16!}{15!} = \square$$

$$\frac{71!}{70!} = \square$$

$$\frac{18!}{17!} - \frac{16!}{15!} - \frac{14!}{13!} + \frac{12!}{11!} = \square$$

3. Counting Spanning Trees in Networks





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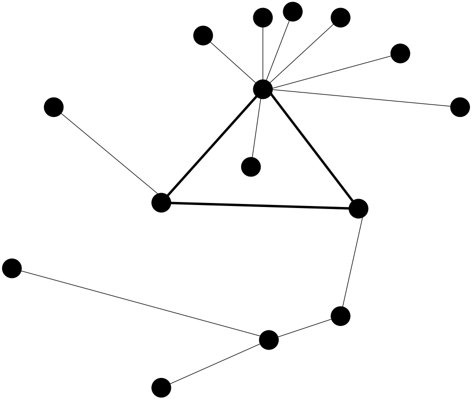

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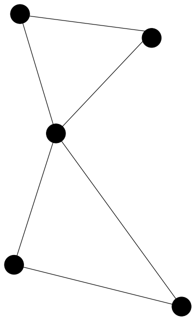

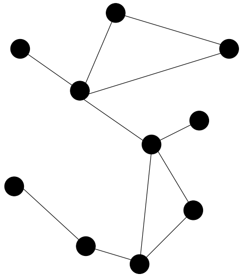

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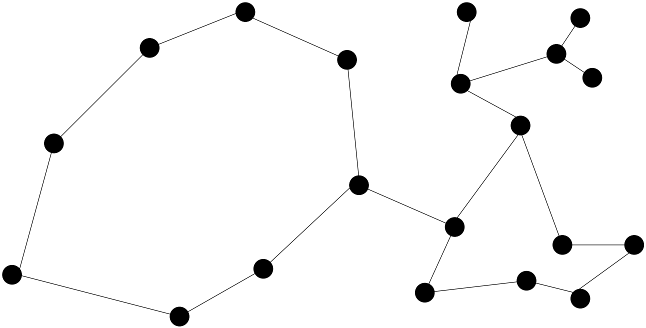

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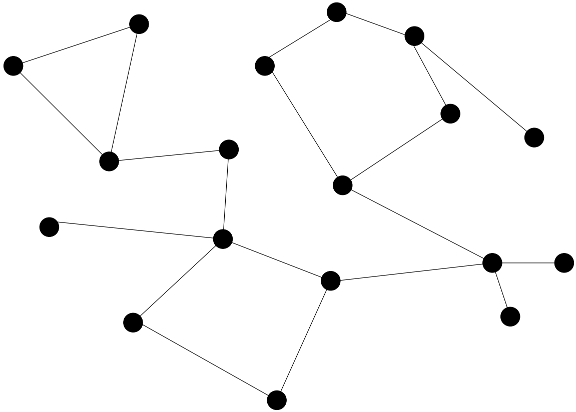


3. Counting Spanning Trees in Networks

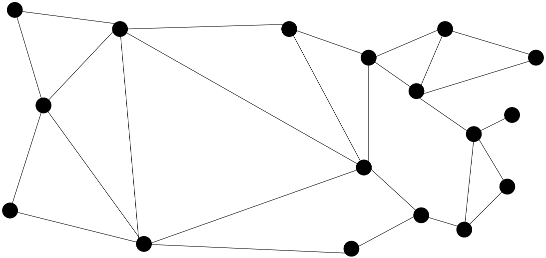
Summary: If a network consists of a bunch of **circuits** attached to each other by **bridges**,

then the **number of spanning trees** is the **product of the lengths of the circuits**.

The other kind of network for which we can easily count trees is a **complete graph** $\mathbf{K_N}$, where there are $\mathbf{N^{N-2}}$ spanning trees. T

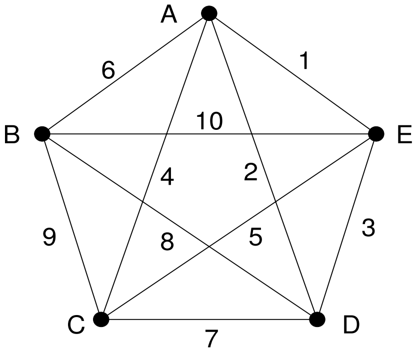
3. Counting Spanning Trees in Networks

Many networks are not of this form — the circuits might overlap.



Here, it is very hard to count the spanning trees.

4. Kruskal's Algorithm



Weights of edges

	A	B	C	D	E
A		6	4	2	1
B	6		9	8	10
C	4	9		7	5
D	2	8	7		3
E	1	10	5	3	