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Definition: The **Traveling Salesman Problem** is the problem of finding a **minimum-weight Hamilton circuit** in K_N.

The Nearest-Neighbor Algorithm

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3. At each stage in your tour, walk to the nearest neighbor that you have not already visited.
4. When you have visited all vertices, return to the starting vertex.

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- Usually, there is no way to know in advance which reference vertex will work the best.
- Once you find a Hamilton circuit, you can start your tour anywhere you want.

Example: Willy's Tour of Australia

Ref. vertex	Hamilton circuit	Weight	
AD	AD,ML,HO,CN,SY,BT,MK,CS,MI,AS,UL,BM,KU,DA,PE,AL,AD	18543	
AL	AL,PE,BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BT,MK,CS,MI,AL	19795	
AS	AS,UL,BM,KU,DA,MI,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,AS	18459	
BT	BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,AL,PE,BT	22113	
BM	BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BT,MK,CS,MI,PE,AL,BM	19148	
CS	CS,MK,BT,SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,PE,AL,CS	22936	
CN	CN,SY,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BT,AL,PE,CN	21149	
DA	DA,KU,BM,UL,AS,MI,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,DA	18543	
HO	HO,ML,CN,SY,BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,HO	20141	
KU	KU,DA,AS,UL,BM,MI,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,KU	18785	
MK	MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,BT,AL,PE,MK	23255	
ML	ML,HO,CN,SY,BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,ML	20141	
MI	MI,AS,UL,BM,KU,DA,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,MI	20877	
PE	PE,AL,BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BT,MK,CS,MI,PE	19148	
SY	SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BT,AL,PE,SY	21049	(NNA)
UL	UL,AS,MI,CS,MK,BT,SY,CN,ML,HO,AD,BM,KU,DA,PE,AL,UL	20763	

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BT	BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,ML,HO,CN,SY,AL,PE,BT	22113	
BM	BM,KU,DA,AS,UL,AD,ML,HO,CN,SY,BT,MK,CS,MI,PE,AL,BM	19148	
CS	CS,MK,BT,SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,PE,AL,CS	22936	
CN	CN,SY,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BT,AL,PE,CN	21149	
DA	DA,KU,BM,UL,AS,MI,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,DA	18543	
HO	HO,ML,CN,SY,BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,HO	20141	
KU	KU,DA,AS,UL,BM,MI,CS,MK,BT,SY,CN,ML,HO,AD,AL,PE,KU	18785	
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ML	ML,HO,CN,SY,BT,MK,CS,MI,AS,UL,BM,KU,DA,AD,AL,PE,ML	20141	
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SY	SY,CN,ML,HO,AD,AS,UL,BM,KU,DA,MI,CS,MK,BT,AL,PE,SY	21049	(NNA)
UL	UL,AS,MI,CS,MK,BT,SY,CN,ML,HO,AD,BM,KU,DA,PE,AL,UL	20763	

The Repetitive Nearest-Neighbor Algorithm

Using Alice Springs (AS) as the reference vertex yields the best result:

AS $\rightarrow$ UL $\rightarrow$ BM $\rightarrow$ KU $\rightarrow$ DA $\rightarrow$ MI $\rightarrow$ CS $\rightarrow$ MK $\rightarrow$ BT $\rightarrow$ SY $\rightarrow$ CN $\rightarrow$ ML $\rightarrow$ HO $\rightarrow$ AD $\rightarrow$ AL $\rightarrow$ PE $\rightarrow$ AS

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Remember: Willy can still start anywhere he wants!
For instance,

SY $\to$ CN $\to$ ML $\to$ HO $\to$ AD $\to$ AL $\to$ PE $\to$ AS $\to$ UL $\to$ BM $\to$ KU $\to$ DA $\to$ MI $\to$ CS $\to$ MK $\to$ BT $\to$ SY

represents the same Hamilton circuit.

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- Make sure you add exactly two edges at each vertex.
- Don't close the circuit until all vertices are in it.

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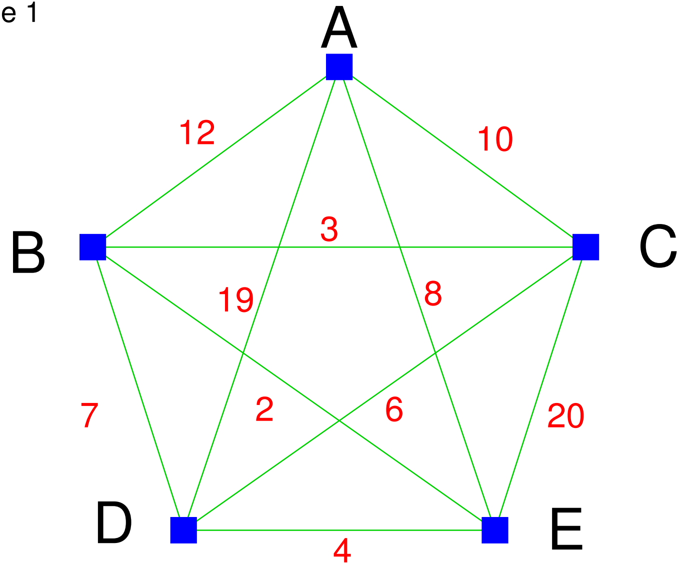
This is called the **Cheapest-Link Algorithm, or CLA**.
Here is an example.

Example 1



Results of Example 1

- Output of RNNA: **BEDCAB** (weight **34**)
- Output of CLA: **ACBEDA** (weight **38**)

- In this example, RNNA produces a better result.

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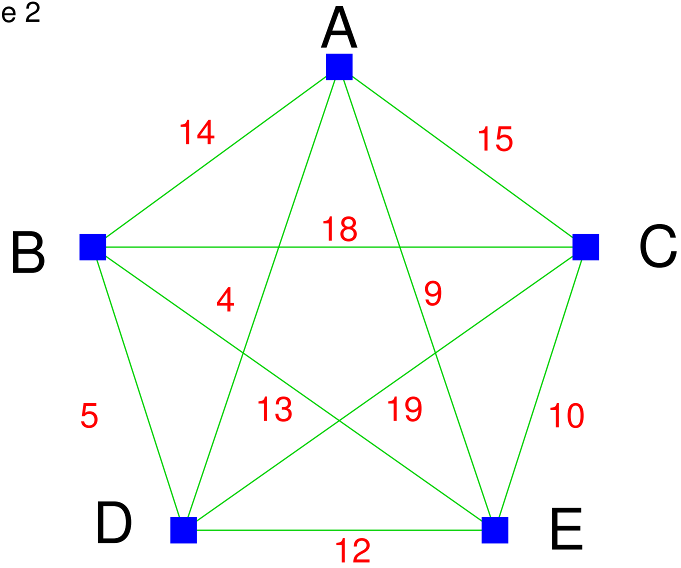
- In fact, neither of these Hamilton circuits is optimal – the optimal one is **EACBDE** (weight **32**).

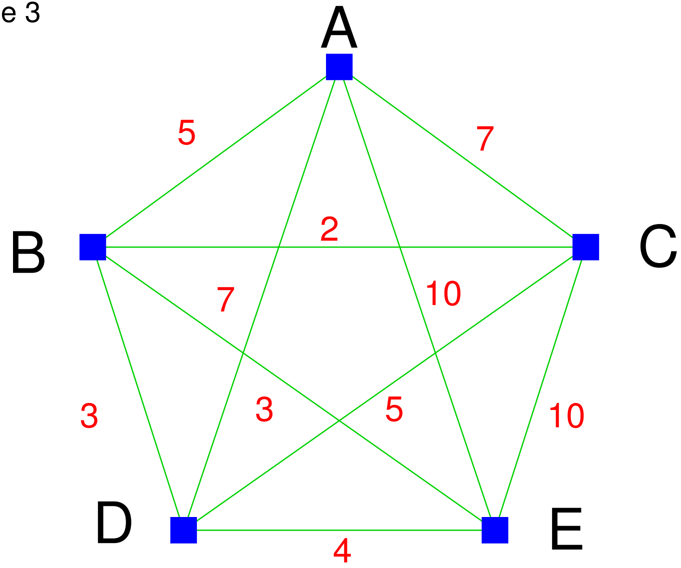
Example 2



Results of Example 2

- RNNA and CLA both output **DAECBD** (weight **46**)
- This happens to be an optimal Hamilton circuit.

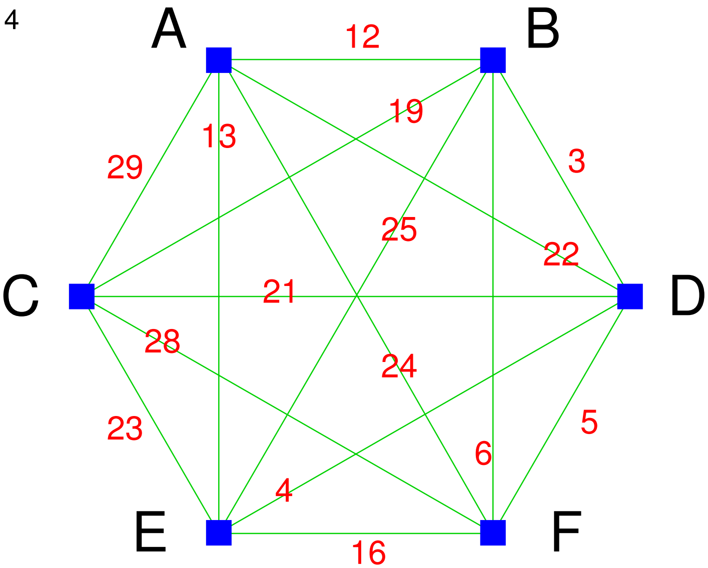
Example 3



Results of Example 3

- Here, the output of both the CLA and the RNNA may depend on how you break ties. (There's no way to know in advance.)

Example 4



Distance table for Example 4

	A	B	C	D	E	F
A		12	29	22	13	24
B	12		19	3	25	6
C	29	19		21	23	28
D	22	3	21		4	5
E	13	25	23	4		16
F	24	6	28	5	16	

Results of Example 4

- Output of RNNA: **FDBAECF** (weight **84**)
- Output of CLA: **ACFBDEA** (weight **83**)

- In this example, CLA produces a better result.

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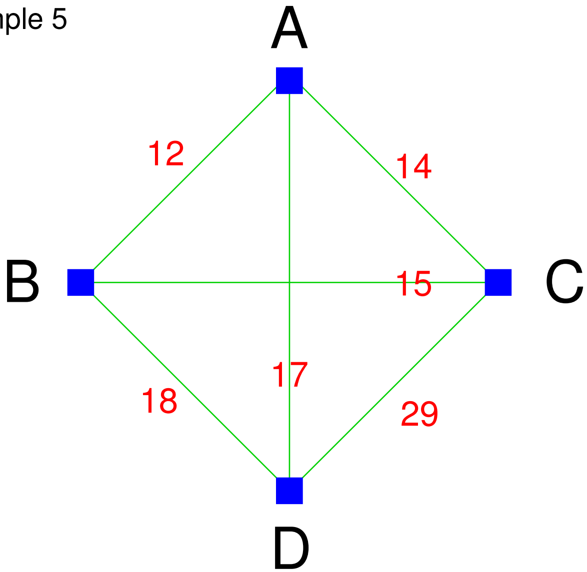
- Output of RNNA: **FDBAECF** (weight **84**)
- Output of CLA: **ACFBDEA** (weight **83**)

- In this example, CLA produces a better result.

- Neither of these Hamilton circuits is optimal – the optimal one is **FBCAEDF** (weight **76**).

Example 5



Results of Example 5

Algorithm	Output	Weight
NNA (A)	**ABCDA**	$12 + 15 + 29 + 17 =$ **73**
NNA (B)	**BACDB**	$12 + 14 + 29 + 18 =$ **73**
NNA (C)	CABDC	= BACDB
NNA (D)	DABCD	= ABCDA
CLA	ABCDA	

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Algorithm	**Output**	**Weight**
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CLA	ABCDA	

- The only other Hamilton circuit in K_4 is **ACBDA**, which has weight $14 + 15 + 18 + 17 =$ **64**.

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NNA (C)	CABDC	= BACDB
NNA (D)	DABCD	= ABCDA
CLA	ABCDA	

- The only other Hamilton circuit in K_4 is **ACBDA**, which has weight $14+15+18+17=$ **64**.
- So both RNNA and CLA give **worst possible answers!**

The Bad News

There is no known algorithm to solve the TSP that is both optimal and efficient.

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- Brute-force is optimal but not efficient.
- NNA, RNNA, and CLA are all efficient but not optimal.