

Coalitions

Coalition: Any set of players.¹

Weight of a coalition: The total number of votes controlled by the players in the coalition; that is, the sum of the weights of individual players in the coalition.

Winning coalition: A coalition whose weight is at least q (enough to pass a motion).

Losing coalition: A coalition whose weight is less than q (not enough to pass a motion).

Grand coalition: The coalition consisting of all players. (This is always a winning coalition.)

¹Tannenbaum (p.48) requires that a coalition contain at least one player. I (JLM) would say that the set with zero players also counts as a coalition.

Coalitions

For example, in the WVS [58; 31, 31, 28, 21, 2, 2]:

- ▶ $\{P_2, P_3, P_5\}$ and $\{P_1, P_4, P_5, P_6\}$ are coalitions (sets of players).

- ▶ $\{P_2, P_3, P_5\}$ is a **winning coalition** because

$$w_2 + w_3 + w_5 = 31 + 28 + 2 = \mathbf{61} \geq \mathbf{58}.$$

- ▶ $\{P_1, P_4, P_5, P_6\}$ is a **losing coalition** because

$$w_1 + w_4 + w_5 + w_6 = 31 + 21 + 2 + 2 = \mathbf{56} < \mathbf{58}.$$

Counting Coalitions

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- ▶ $N = 2$: **4 coalitions:**

$$\{\}, \{P_1\}, \{P_2\}, \{P_1, P_2\}$$

Counting Coalitions

How many different possible coalitions are there?

▶ $N = 1$: **2 coalitions:** $\{\}$ and $\{P_1\}$.

▶ $N = 2$: **4 coalitions:**

$\{\}$, $\{P_1\}$, $\{P_2\}$, $\{P_1, P_2\}$

▶ $N = 3$: **8 coalitions:**

$\{\}$, $\{P_1\}$, $\{P_2\}$, $\{P_1, P_2\}$,
 $\{P_3\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$

Counting Coalitions

How many different possible coalitions are there?²

If there are N players, there are 2^N coalitions.

N	2^N	N	2^N
0	1	6	64
1	2	7	128
2	4	8	256
3	8	9	512
4	16	10	1024
5	32	20	1048576

²This is if you count the coalition $\{\}$. If you follow Tannenbaum and require that a coalition contain at least one player, then the formula would be $2^N - 1$ instead of 2^N .

Coalition Tables

Coalition table: A list of all possible coalitions that indicates which are winning and which are losing.

Example 1: [8; 5, 5, 4].

Coalition	Total votes	Winning/losing
$\{\}$	0	Losing
$\{P_1\}$	5	Losing
$\{P_2\}$	5	Losing
$\{P_3\}$	4	Losing
$\{P_1, P_2\}$	10	Winning
$\{P_1, P_3\}$	9	Winning
$\{P_2, P_3\}$	9	Winning
$\{P_1, P_2, P_3\}$	14	Winning

Coalition Tables

Coalition table: A list of all possible coalitions that indicates which ones are winning and which ones are losing.

Example 2: [10; 5, 5, 4].

Coalition	Total votes	Winning/losing
$\{\}$	0	Losing
$\{P_1\}$	5	Losing
$\{P_2\}$	5	Losing
$\{P_3\}$	4	Losing
$\{P_1, P_2\}$	10	Winning
$\{P_1, P_3\}$	9	Losing
$\{P_2, P_3\}$	9	Losing
$\{P_1, P_2, P_3\}$	14	Winning

Coalition Tables

Coalition table: A list of all possible coalitions that indicates which ones are winning and which ones are losing.

Example 3: [12; 5, 5, 4].

Coalition	Total votes	Winning/losing
$\{\}$	0	Losing
$\{P_1\}$	5	Losing
$\{P_2\}$	5	Losing
$\{P_3\}$	4	Losing
$\{P_1, P_2\}$	10	Losing
$\{P_1, P_3\}$	9	Losing
$\{P_2, P_3\}$	9	Losing
$\{P_1, P_2, P_3\}$	14	Winning

Equivalence of Weighted Voting Systems

We can now tell when two weighted voting systems represent the same allocation of power:

when the same coalitions are winning in both systems.

For example, let's compare two apparently different WVS's:

$[10; 5, 5, 4]$ and $[60; 44, 22, 11]$.

Equivalence of Weighted Voting Systems

Coalition	[10; 5, 5, 4]		[60; 44, 22, 11]	
	Total	W/L	Total	W/L
$\{\}$				
$\{P_1\}$				
$\{P_2\}$				
$\{P_3\}$				
$\{P_1, P_2\}$				
$\{P_1, P_3\}$				
$\{P_2, P_3\}$				
$\{P_1, P_2, P_3\}$				

Equivalence of Weighted Voting Systems

Coalition	[10; 5, 5, 4]		[60; 44, 22, 11]	
	Total	W/L	Total	W/L
$\{\}$	0	Losing		
$\{P_1\}$	5	Losing		
$\{P_2\}$	5	Losing		
$\{P_3\}$	4	Losing		
$\{P_1, P_2\}$	10	Winning		
$\{P_1, P_3\}$	9	Losing		
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$\{P_1, P_2, P_3\}$	14	Winning		

Equivalence of Weighted Voting Systems

Coalition	[10; 5, 5, 4]		[60; 44, 22, 11]	
	Total	W/L	Total	W/L
$\{\}$	0	Losing	0	Losing
$\{P_1\}$	5	Losing	44	Losing
$\{P_2\}$	5	Losing	22	Losing
$\{P_3\}$	4	Losing	11	Losing
$\{P_1, P_2\}$	10	Winning	66	Winning
$\{P_1, P_3\}$	9	Losing	55	Losing
$\{P_2, P_3\}$	9	Losing	33	Losing
$\{P_1, P_2, P_3\}$	14	Winning	77	Winning

Equivalence of Weighted Voting Systems

Coalition	[10; 5, 5, 4]		[60; 44, 22, 11]	
	Total	W/L	Total	W/L
$\{\}$	0	Losing	0	Losing
$\{P_1\}$	5	Losing	44	Losing
$\{P_2\}$	5	Losing	22	Losing
$\{P_3\}$	4	Losing	11	Losing
$\{P_1, P_2\}$	10	Winning	66	Winning
$\{P_1, P_3\}$	9	Losing	55	Losing
$\{P_2, P_3\}$	9	Losing	33	Losing
$\{P_1, P_2, P_3\}$	14	Winning	77	Winning

These two WVS's are equivalent!

Dummies

Question: How can we detect dummies mathematically?

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Answer: Player P is a dummy if there is **no winning coalition that P can turn into a losing coalition by deserting.**

Equivalently, P is a dummy if there is **no losing coalition that P can turn into a winning coalition by joining.**

(A dummy can still be part of a winning coalition! But a dummy's presence or absence **makes no difference** to whether or not the coalition is winning or losing.)

Dummies

Example: [10; 5, 5, 4].

- ▶ The only winning coalitions are $\{P_1, P_2\}$ and $\{P_1, P_2, P_3\}$ (the grand coalition)
- ▶ The only winning coalition that includes P_3 is the grand coalition...
- ▶ ...and **removing P_3 still results in a winning coalition**, namely $\{P_1, P_2\}$.
- ▶ Therefore, **P_3 is a dummy.**

Dummies

Example: [12; 5, 5, 4].

- ▶ Now the only winning coalition is $\{P_1, P_2, P_3\}$ (the grand coalition).
- ▶ Removing P_3 from the grand coalition results in a **losing coalition**, namely $\{P_1, P_2\}$.
- ▶ Therefore, **P_3 is not a dummy.**

Dummies

Example: $[8; 5, 5, 4]$.

- ▶ There are four winning coalitions: $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, and the grand coalition $\{P_1, P_2, P_3\}$.
- ▶ Removing P_3 from the winning coalition $\{P_1, P_3\}$ results in the losing coalition $\{P_1\}$.
- ▶ Therefore, **P_3 is not a dummy.**

Measuring Power

Question: How do we measure power using mathematics?

Idea: Measure how often each player in a WVS controls the balance of power.

That is, for each player P , what is the likelihood that (a) P is part of the winning coalition and that (b) P 's *support is necessary for that coalition to be a winning one?*

Such a player is called **critical** for that winning coalition.

Critical Players

Example 1: [11; 7, 5, 4].

The winning coalitions and their weights are:

$$\underbrace{\{P_1, P_2\}}_{\text{weight 12}}$$

$$\underbrace{\{P_1, P_3\}}_{\text{weight 11}}$$

$$\underbrace{\{P_1, P_2, P_3\}}_{\text{weight 16}}$$

- ▶ In $\{P_1, P_2\}$, the participation of P_2 is necessary: if P_2 leaves the coalition, it becomes a losing one.
- ▶ In $\{P_1, P_2, P_3\}$, the participation of P_2 is not necessary: if P_2 leaves, then the coalition is still winning.

P_2 is **critical** for $\{P_1, P_2\}$, but not for $\{P_1, P_2, P_3\}$.

Critical Counts and the Banzhaf Power Index

Example 1: [11; 7, 5, 4].

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	$7 + 5 = 12$	P_1, P_2
$\{P_1, P_3\}$	$7 + 4 = 11$	P_1, P_3
$\{P_1, P_2, P_3\}$	$7 + 5 + 4 = 16$	P_1

Critical count = # of coalitions for which a player is critical.

The critical count for P_1 is 3;

the critical counts for P_2 and P_3 are both 1.

Critical Counts and the Banzhaf Power Index

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	$7 + 5 = 12$	P_1, P_2
$\{P_1, P_3\}$	$7 + 4 = 11$	P_1, P_3
$\{P_1, P_2, P_3\}$	$7 + 5 + 4 = 16$	P_1

Player	Critical Count
P_1	3
P_2	1
P_3	1
Total	5

Critical Counts and the Banzhaf Power Index

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	$7 + 5 = 12$	P_1, P_2
$\{P_1, P_3\}$	$7 + 4 = 11$	P_1, P_3
$\{P_1, P_2, P_3\}$	$7 + 5 + 4 = 16$	P_1

Player	Critical Count	Banzhaf power index
P_1	3	$3/5$
P_2	1	$1/5$
P_3	1	$1/5$
Total	5	

Measuring Power, Again

Idea: We can measure a player's power by determining how often that player is critical for some coalition.

I.e., how often does each player's name appear in the last column of the table?

- ▶ Lionel Penrose (1946): "The Elementary Statistics of Majority Voting"
- ▶ John Banzhaf (1965): "Weighted Voting Doesn't Work"
- ▶ James Coleman (1971): "Control of Collectives and the Power of a Collectivity to Act"

Banzhaf's Article

Banzhaf's article "Weighted Voting Doesn't Work" (Rutgers Law Review, vol. 19, 1965)

"In almost all cases weighted voting does not do the one thing which both its supporters and opponents assume that it does . . . *voting power is not proportional to the number of votes a legislator may cast.*"

"The purpose of this paper is neither to attack nor defend weighted voting *per se*. As with any objective mathematical analysis, its intent is only to explain the effects which necessarily follow once the mathematical model and the rules of its operation are established."

Banzhaf's Article

Banzhaf analyzed the Nassau County Board of Supervisors — the weighted voting system [58; 31, 31, 28, 21, 2, 2].

Banzhaf showed that, as we have seen, players P_4 , P_5 and P_6 are dummies.

The article “Weighted Voting Doesn't Work” became the basis for a series of lawsuits (not a frequent achievement for an article on mathematics!)

Calculating Banzhaf Power Indices

For any weighted voting system with N players:

1. Find all the winning coalitions.
2. For each one, determine the critical players.
3. Count the number of times that each player P_i is critical. Call this number B_i .
4. The **Banzhaf power index** of P_i is

$$\beta_i = \frac{B_i}{B_1 + B_2 + \cdots + B_N}$$

The **Banzhaf power distribution** of the weighted voting system is the complete list

$$(\beta_1, \beta_2, \dots, \beta_N).$$

Banzhaf Power: Example 1

Example 1: The weighted voting system $[8; 5, 5, 4]$.

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	$5 + 5 = 10$	P_1, P_2
$\{P_1, P_3\}$	$5 + 4 = 9$	P_1, P_3
$\{P_2, P_3\}$	$5 + 4 = 9$	P_2, P_3
$\{P_1, P_2, P_3\}$	$5 + 5 + 4 = 14$	None

Banzhaf Power: Example 1

- ▶ Each of the three players has the same critical count:

$$B_1 = B_2 = B_3 = 2.$$

- ▶ Thus every player has the same Banzhaf power index. The sum of all critical counts is $T = 6$, so

$$\beta_1 = \beta_2 = \beta_3 = \frac{2}{6} = \frac{1}{3}.$$

- ▶ Therefore, the **Banzhaf power distribution** is

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).$$

- ▶ This confirms that all players have equal power!

Banzhaf Power: Example 2

Example 2: The weighted voting system [10; 5, 5, 4].

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	10	★
$\{P_1, P_2, P_3\}$	14	

Banzhaf Power: Example 2

Example 2: The weighted voting system $[10; 5, 5, 4]$.

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	10	P_1, P_2
$\{P_1, P_2, P_3\}$	14	P_1, P_2

Banzhaf Power: Example 2

Example 2: The weighted voting system $[10; 5, 5, 4]$.

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	10	P_1, P_2
$\{P_1, P_2, P_3\}$	14	P_1, P_2

Player	B_i	Banzhaf power index
P_1	2	$2/4 = 1/2$
P_2	2	$2/4 = 1/2$
P_3	0	0
Total	$T = 4$	

Banzhaf Power: Example 2

Example 2: The weighted voting system $[10; 5, 5, 4]$.

Winning Coalition	Weight	Critical Players
$\{P_1, P_2\}$	10	P_1, P_2
$\{P_1, P_2, P_3\}$	14	P_1, P_2

Player	B_i	Banzhaf power index
P_1	2	$2/4 = 1/2$
P_2	2	$2/4 = 1/2$
P_3	0	0
Total	$T = 4$	

← **Dummy!**

Banzhaf Power: Example 2

Conclusion: In the WVS [10; 5, 5, 4], the Banzhaf power distribution is

$$(\beta_1, \beta_2, \beta_3) = (1/2, 1/2, 0).$$

Therefore, player P_3 is a dummy.

A player is a dummy if and only if his/her Banzhaf power index is 0.

Banzhaf Power: Example 3

Example 3: The weighted voting system [12; 5, 5, 4]

Winning Coalition	Weight	Critical Players
$\{P_1, P_2, P_3\}$	$5 + 5 + 4 = 14$	P_1, P_2, P_3

- ▶ All players have the same critical count:

$$B_1 = B_2 = B_3 = 1.$$

- ▶ So all players have the same Banzhaf power index:

$$\beta_1 = \beta_2 = \beta_3 = 1/3.$$

Banzhaf Power: Examples 1 and 3

- ▶ In both $[8; 5, 5, 4]$ and $[12; 5, 5, 4]$, the Banzhaf power distribution is

$$(\beta_1, \beta_2, \beta_3) = (1/3, 1/3, 1/3).$$

- ▶ Therefore, in each system, all players have the same power.
- ▶ But that **does not necessarily mean** that the voting systems are equivalent.

(In the first case, a motion requires two players to agree; in the second case, unanimous agreement is required.)

Banzhaf Power: Example 4

Example 4: Weighted voting system [22; 26, 5, 5, 4].

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- ▶ There are lots of winning coalitions — namely, those (and only those) coalitions that contain player P_1 (“Mom”).

Banzhaf Power: Example 4

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- ▶ There are lots of winning coalitions — namely, those (and only those) coalitions that contain player P_1 (“Mom”).
- ▶ In each winning coalition, Mom is a critical player, but no one else is.

Banzhaf Power: Example 4

Example 4: Weighted voting system [22; 26, 5, 5, 4].

- ▶ There are lots of winning coalitions — namely, those (and only those) coalitions that contain player P_1 (“Mom”).
- ▶ In each winning coalition, Mom is a critical player, but no one else is.
- ▶ Therefore, Mom’s Banzhaf power index is 1 (= 100%) and everyone else’s Banzhaf power index is 0.

Banzhaf Power: Example 4

Example 4: Weighted voting system [22; 26, 5, 5, 4].

- ▶ There are lots of winning coalitions — namely, those (and only those) coalitions that contain player P_1 (“Mom”).
- ▶ In each winning coalition, Mom is a critical player, but no one else is.
- ▶ Therefore, Mom’s Banzhaf power index is 1 (= 100%) and everyone else’s Banzhaf power index is 0.

A player is a dictator if and only if his/her Banzhaf power index is 100%.

Banzhaf Power: Summary

- ▶ The Banzhaf power index (BPI) of a player measures how often that player is critical for some winning coalition.

$$\beta_i = \frac{B_i}{T} = \frac{\text{critical count for player } P_i}{\text{sum of critical counts for all players}}$$

- ▶ The BPI of a dummy is always 0 (= 0%).
- ▶ The BPI of a dictator is always 1 (= 100%).
- ▶ If two players have the same Banzhaf power index, then they really do have the same amount of power.

Banzhaf Power: Example 5

Example 5: In the weighted voting system $[10; 8, 4, 2, 1]$, what is the Banzhaf power distribution? ★

Banzhaf Power: Example 5

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Winning Coalition	Critical Players
$\{P_1, P_2\}$	P_1, P_2
$\{P_1, P_3\}$	P_1, P_3
$\{P_1, P_2, P_3\}$	P_1, P_2
$\{P_1, P_2, P_4\}$	P_1, P_2
$\{P_1, P_3, P_4\}$	P_1
$\{P_1, P_2, P_3, P_4\}$	P_1

Banzhaf Power: Example 5

Example 5: In the weighted voting system $[10; 8, 4, 2, 1]$, what is the Banzhaf power distribution? ★

Winning Coalition	Critical Players
$\{P_1, P_2\}$	P_1, P_2
$\{P_1, P_3\}$	P_1, P_3
$\{P_1, P_2, P_3\}$	P_1, P_2
$\{P_1, P_2, P_4\}$	P_1, P_2
$\{P_1, P_3, P_4\}$	P_1
$\{P_1, P_2, P_3, P_4\}$	P_1

$$B_1 = 6$$

$$B_2 = B_3 = 2$$

$$B_4 = 0$$

$$T = 10$$

Banzhaf Power: Example 5

For the weighted voting system $[10; 8, 4, 2, 1]$,
the Banzhaf power distribution is

$$(6/10, 2/10, 2/10, 0) = (3/5, 1/5, 1/5, 0).$$

- ▶ P_1 has veto power (he is critical for every winning coalition).
- ▶ P_2 has the same amount of power as P_3 .
- ▶ P_4 is a dummy (her Banzhaf power index is 0).

Example: The UN Security Council

The UN Security Council consists of

- ▶ 5 permanent members: China, France, Russia, United Kingdom, USA
- ▶ 10 rotating members (currently: Bosnia/Herzegovina, Brazil, Gabon, Lebanon, Nigeria, Colombia, Germany, India, Portugal, South Africa)

Passing a motion requires at least nine votes, including all five of the permanent members.

What is the Banzhaf power distribution?

(There must be a better way of doing this than having to write out all $2^{15} = 32,768$ coalitions!)

Example: The UN Security Council

Observations:

- ▶ We don't need to assign numerical weights — all we need to do is figure out which coalitions are winning and which ones are losing.
- ▶ Each winning coalition must contain **all five permanent members**, along with **at least four rotating members**.
- ▶ Every permanent member has the same power as every other permanent member.
- ▶ Every rotating member has the same power as every other rotating member.

Example: The UN Security Council

Permanent Members

- ▶ Every permanent member (say, Russia) is critical for **every winning coalition**.
- ▶ Therefore, the critical count for Russia is just the total number of winning coalitions, which is **the number of ways to choose at least four of the ten rotating members**.
- ▶ Using permutations and combinations (coming soon!), this number can be calculated as **848**.

Example: The UN Security Council

Rotating Members

- ▶ Each rotating member (say, Portugal) is critical for a winning coalition **only if there are exactly three other rotating members**.
- ▶ Therefore, the critical count for Portugal is **the number of ways to choose exactly three of the nine other rotating members**.
- ▶ Using permutations and combinations, this number can be calculated as **84**.

Example: The UN Security Council

5 permanent members, each with critical count 848

10 rotating members, each with critical count 84

Example: The UN Security Council

5 permanent members, each with critical count 848

10 rotating members, each with critical count 84

Banzhaf power index of each permanent member:

$$\frac{848}{(5 \times 848) + (10 \times 84)} = \frac{848}{5080} \approx 0.1669 = \mathbf{16.69\%}$$

Example: The UN Security Council

5 permanent members, each with critical count 848

10 rotating members, each with critical count 84

Banzhaf power index of each permanent member:

$$\frac{848}{(5 \times 848) + (10 \times 84)} = \frac{848}{5080} \approx 0.1669 = \mathbf{16.69\%}$$

Banzhaf power index of each rotating member:

$$\frac{84}{(5 \times 848) + (10 \times 84)} = \frac{84}{5080} \approx 0.0165 = \mathbf{1.65\%}$$