The Notorious Four-Color Problem

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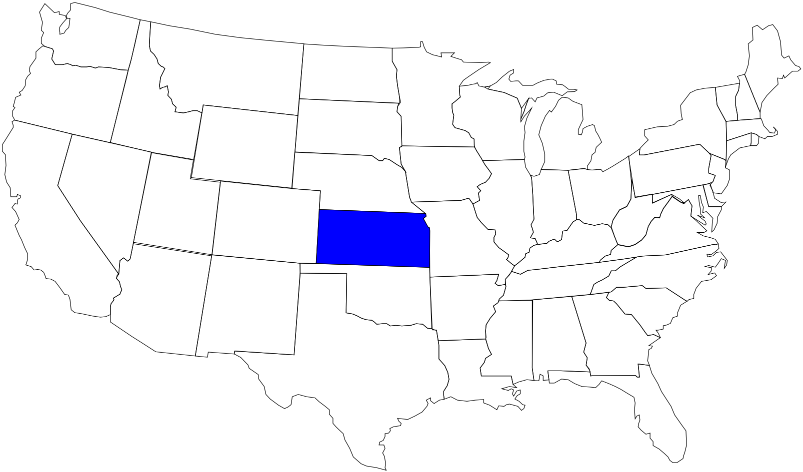
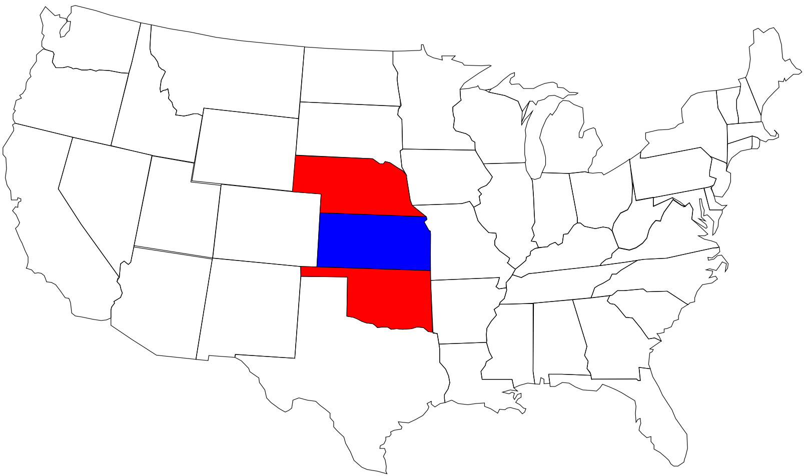
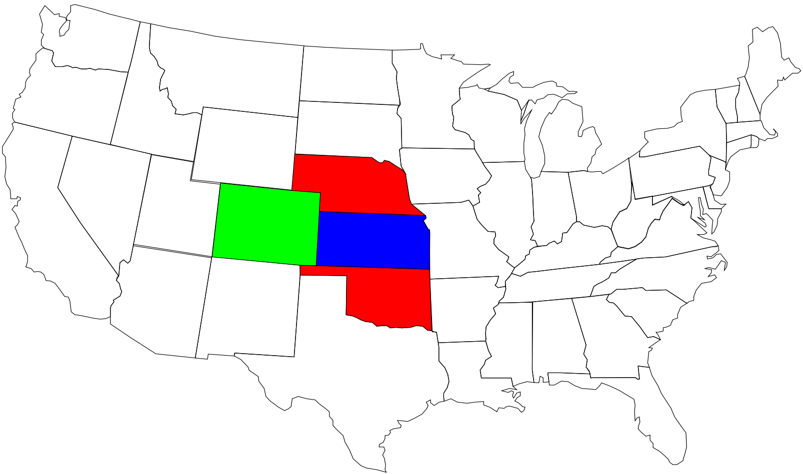
June 5, 2013

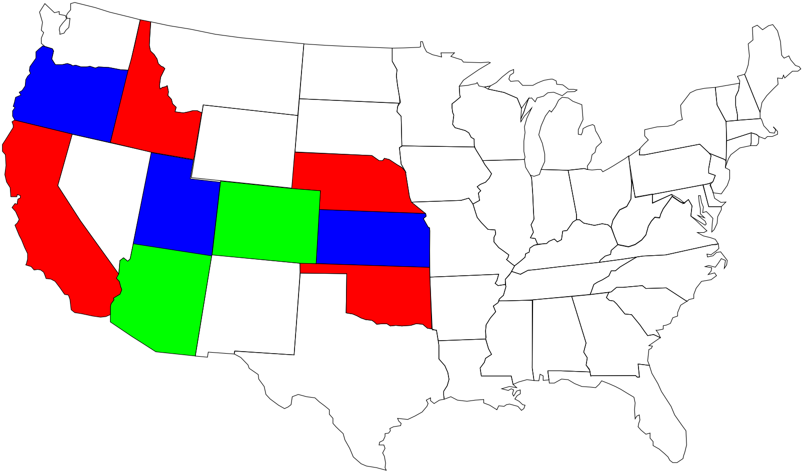
The Map-Coloring Problem

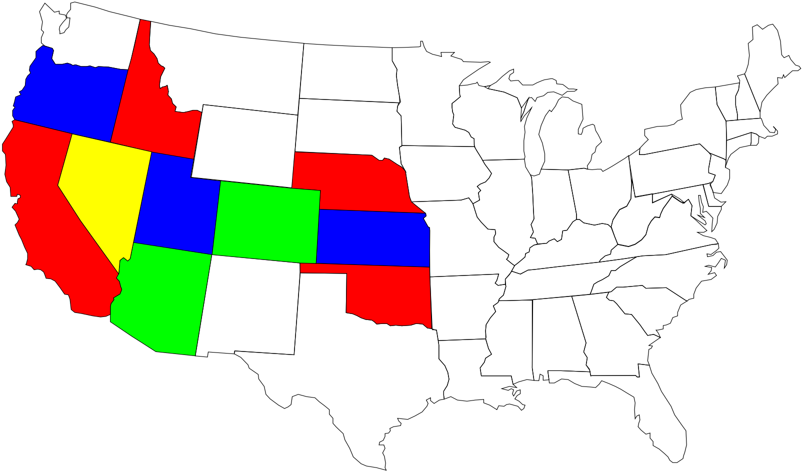
Question: How many colors are required to color a map of the United States so that no two adjacent regions are given the same color?

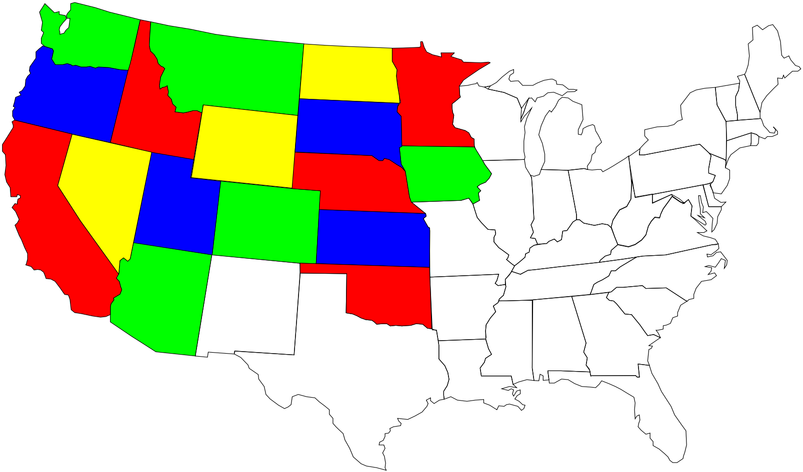


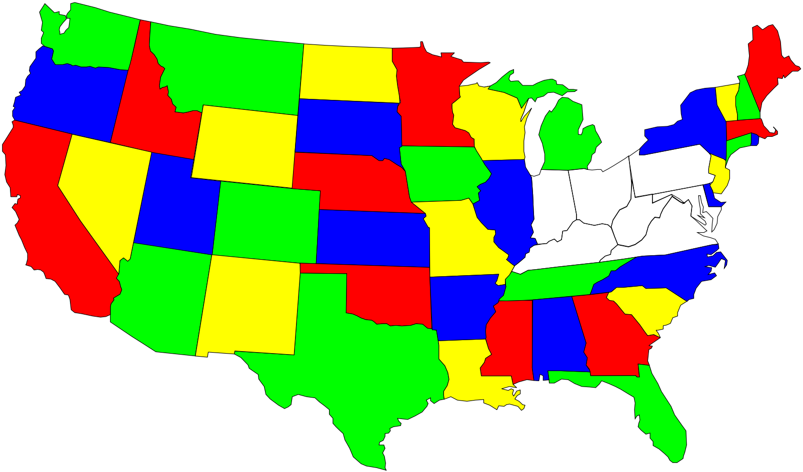


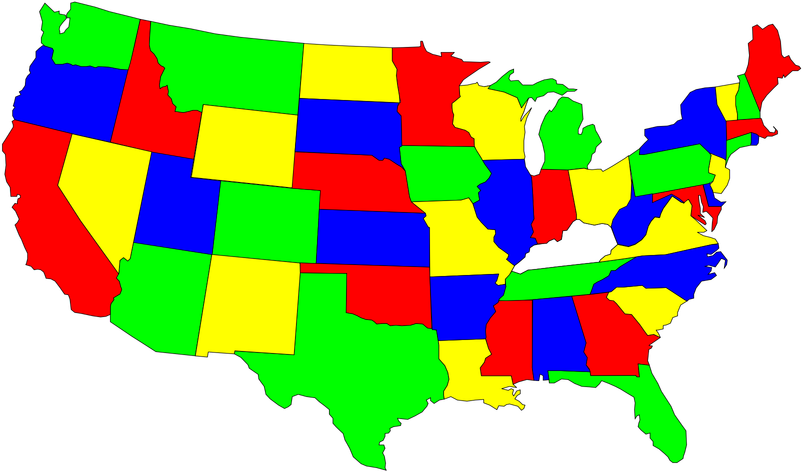


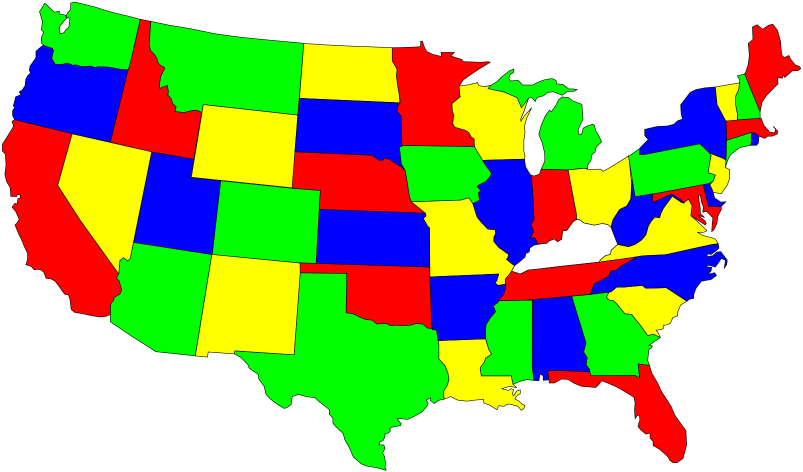


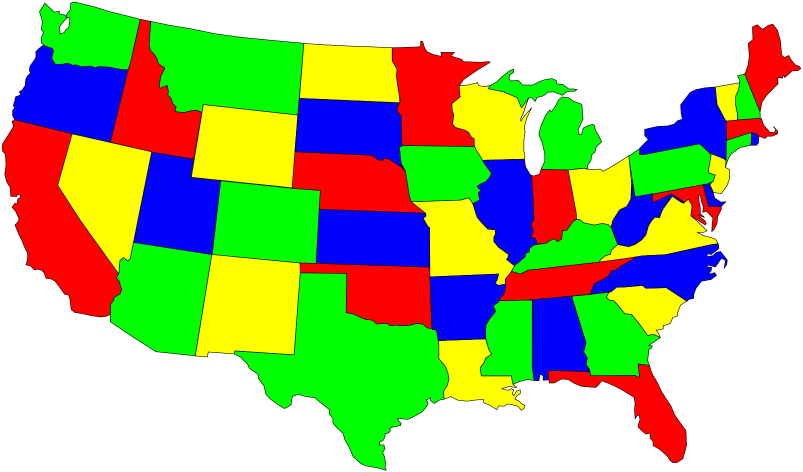












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Answer: Four colors are enough. Three are not enough.

The Four-Color Problem: Is there some map that requires five colors?

In order to give a negative answer, you have to show that *every* map — no matter how cleverly constructed — can be colored with 4 or fewer colors.

The History of the Four-Color Theorem

1852: Student Francis Guthrie notices that four colors suffice to color a map of the counties of England.

Guthrie poses the Four-Color Problem to his brother Frederick, a student of Augustus De Morgan (a big shot of 19th-century British mathematics).

De Morgan likes the problem and mentions it to others.

The History of the Four-Color Theorem

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- 1890: Percy John Heawood shows that Kempe's proof was wrong.
- 1891: Julius Petersen shows that Tait's proof was wrong.
- 20th century: Many failed attempts to (dis)prove the 4CT. Some lead to interesting discoveries; many don't.

The History of the Four-Color Theorem

Mathematician H.S.M. Coxeter:

> *Almost every mathematician must have experienced one glorious night when he thought he had discovered a proof, only to find in the morning that he had fallen into a similar trap.*

Mathematician Underwood Dudley:

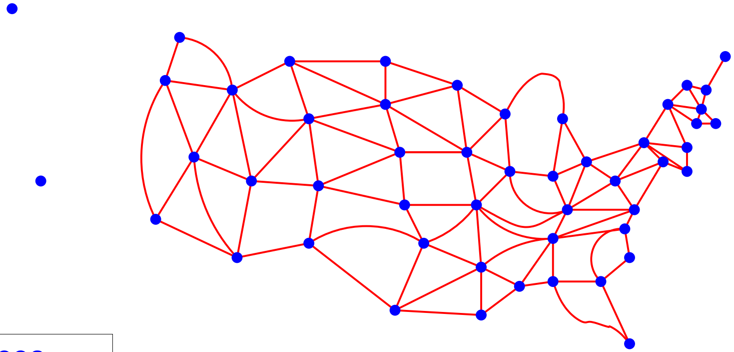
> *The four-color conjecture was easy to state and easy to understand, no large amount of technical mathematics is needed to attack it, and errors in proposed proofs are hard to see, even for professionals; what an ideal combination to attract cranks!*

The History of the Four-Color Theorem

- 1976: Kenneth Appel and Wolfgang Haken prove the 4CT. Their proof relies on checking a large number of cases by computer, sparking ongoing debate over what a proof really is.
- 1997: N. Robertson, D.P. Sanders, P.D. Seymour, and R. Thomas improve Appel and Haken's methods to reduce the number of cases (but still rely on computer assistance).
- 2005: Georges Gonthier publishes a "formal proof" (automating not just the case-checking, but the proof process itself).

Graphs

A graph consists of a collection of vertices connected by edges.



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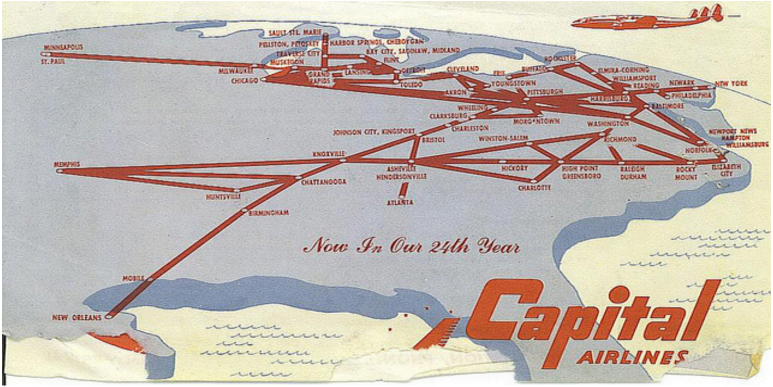

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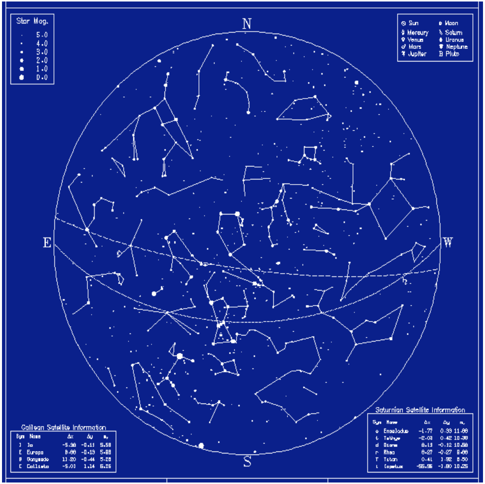

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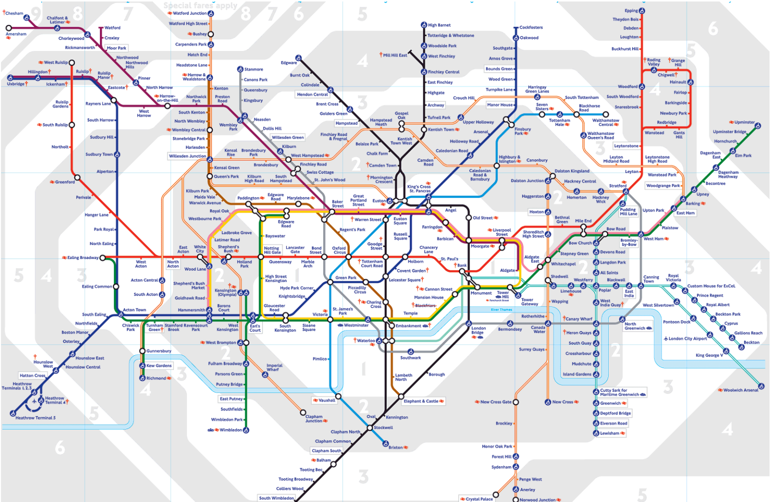

Graphs

The vertices and edges of a graph do not have to be points and curves.

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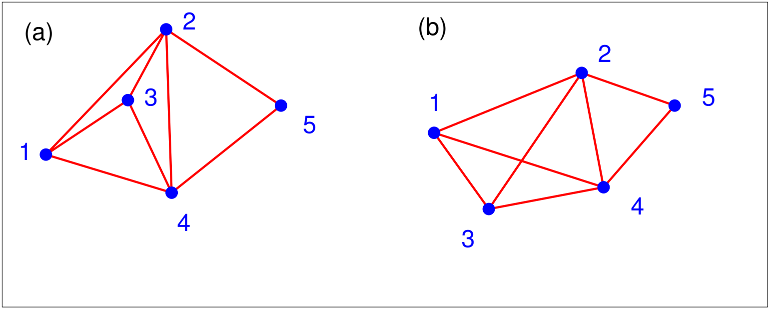
- Facebook: vertices = people, edges = friendship
- WWW: vertices = web pages, edges = links
- Chess: vertices = positions, edges = possible moves

Graphs vs. Drawings

A graph remains the same no matter how you draw it.



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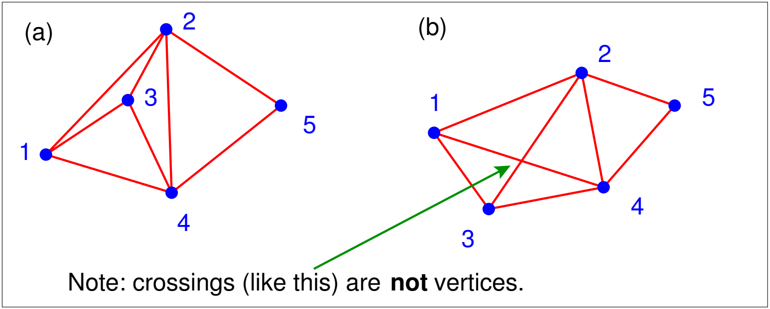

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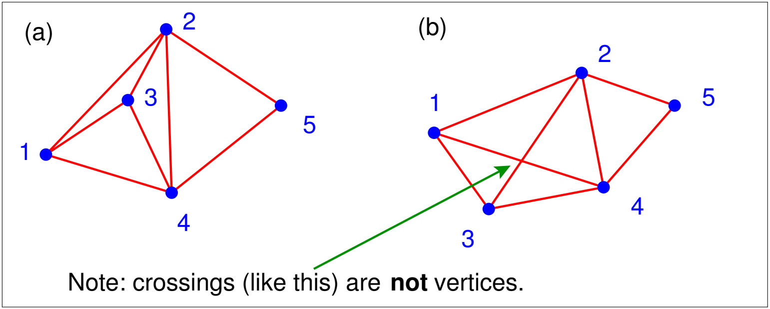

These drawings represent the same graph (e.g., vertex 5 has neighbors 2,4 in both cases). All that matters is which pairs of vertices are connected.

Planar Graphs

A graph is planar if its vertices and edges **can** be drawn as points and line segments with no crossings.

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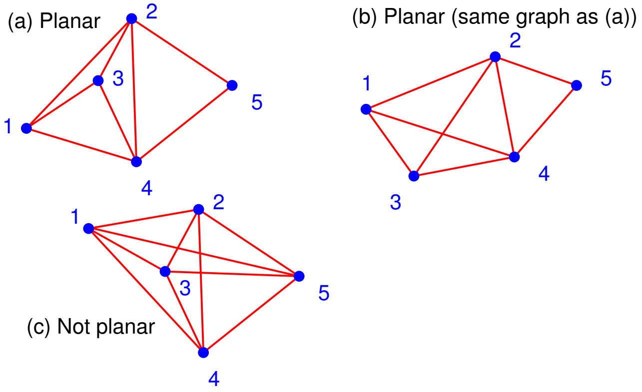

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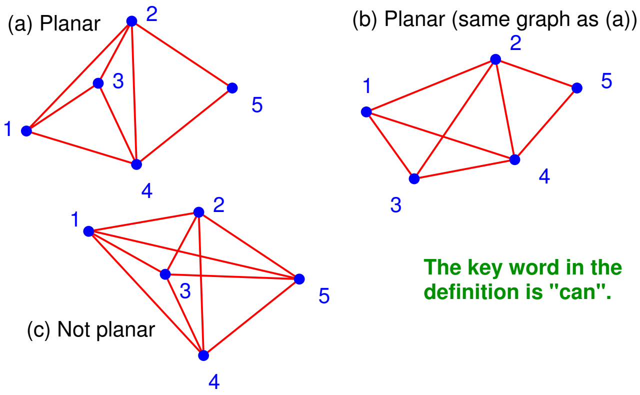
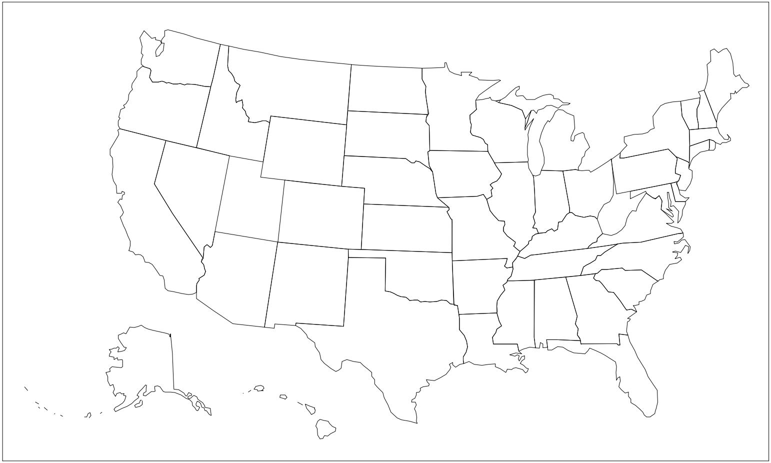


The key word in the definition is "can".

Graphs from Maps

Every map can be modeled as a planar graph.
Vertices represent regions; edges represent common borders.



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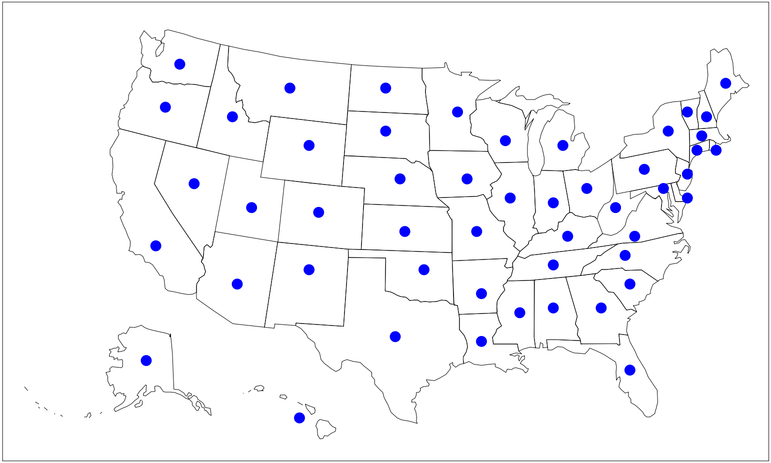

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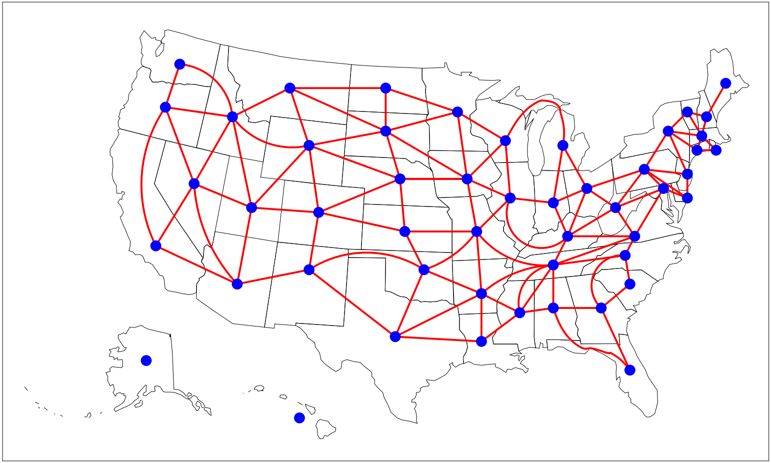

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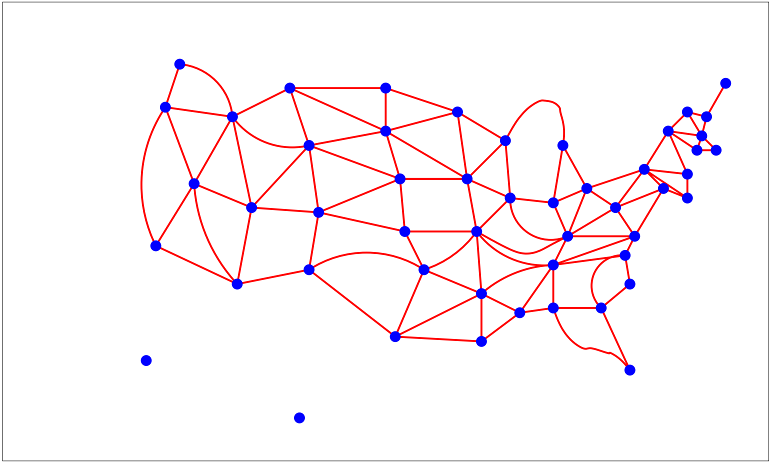

Graph Coloring

Example: You are a kindergarten teacher. You want to assign each child a table to sit at. However, there are certain pairs of kids who shouldn't sit together.

How many tables are you going to need?

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How many tables are you going to need?

This is a graph theory problem!

vertices = children
edges = pairs of kids to keep separate
colors = tables

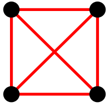
Graph Coloring and the Chromatic Number

A proper coloring of G is an assignment of colors to the vertices of G such that every two vertices connected by an edge must receive different colors.

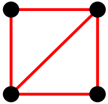
The chromatic number $\chi(G)$ of G is the minimum number of colors needed for a proper coloring.

"G is k-colorable" means "the chromatic number of G is k or less".

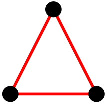
Graph Coloring and the Chromatic Number



$\chi = 4$



$\chi = 3$



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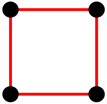
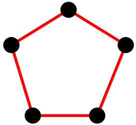
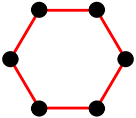


$\chi = 2$



$\chi = 3$



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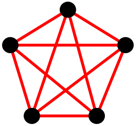
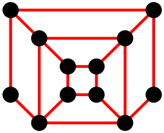
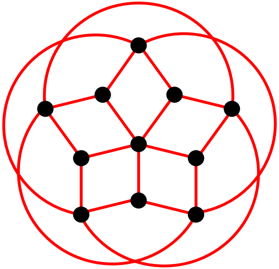


$\chi = 5$



$\chi = 2$



$\chi = 4$

Graph Coloring and the Chromatic Number

A proper coloring of G is an assignment of colors to the vertices of G such that every two vertices connected by an edge must receive different colors.

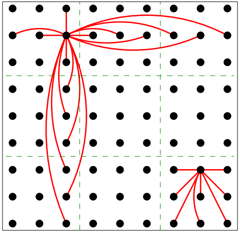
The chromatic number $\chi(G)$ is the minimum number of colors needed for a proper coloring.

Important Note: The chromatic number is not necessarily the same as the maximum number of mutually connected vertices. (For example, a graph can have chromatic number 3 even if it has no triangles.)

Graph Coloring

Consider a graph whose vertices are a 9×9 grid of points.

Two vertices are joined by an edge if they are in the same row, column, or 3×3 subregion.



Not all edges shown – each vertex has 20 neighbors.

You are given a partial proper coloring and told to extend it to all vertices. Does this sound familiar?

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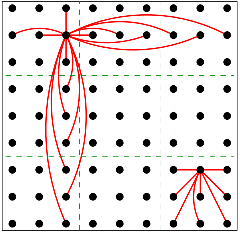


Not all edges shown – each vertex has 20 neighbors.

You are given a partial proper coloring and told to extend it to all vertices. Does this sound familiar? It's a Sudoku puzzle!

Graph Coloring

A proper coloring of G is an assignment of colors to the vertices of G such that every two vertices connected by an edge must receive different colors.

The chromatic number $\chi(G)$ of G is the minimum number of colors needed for a proper coloring.

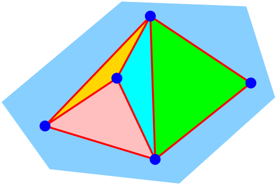
The Four-Color Problem: Does every planar graph have chromatic number 4 or less?

Interlude: What Is A Proof?

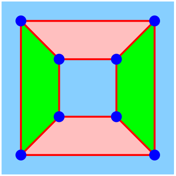
- What is a mathematical proof?
  - A logical argument that relies on commonly accepted axioms and rules of inference (e.g., "if $a = b$ and $b = c$, then $a = c$").
- When is a proof correct?
  - The standard of proof is very high in mathematics (not just "beyond a reasonable doubt", but beyond **any** doubt)
- Who gets to decide whether a proof is correct?
- Are some proofs better than others?

Back to Graph Theory: Faces of Planar Graphs

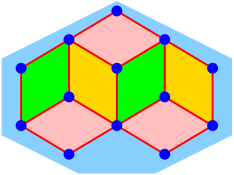
Planar graphs have faces as well as edges and vertices. The faces are the areas between the edges.



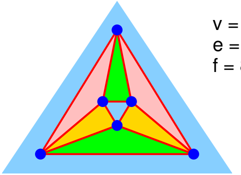
v = 5
e = 8
f = 5



v = 8
e = 12
f = 6



v = 13
e = 19
f = 8



v = 6
e = 12
f = 8

Euler's Formula

Theorem (Euler's Formula)

Let G be any planar graph.

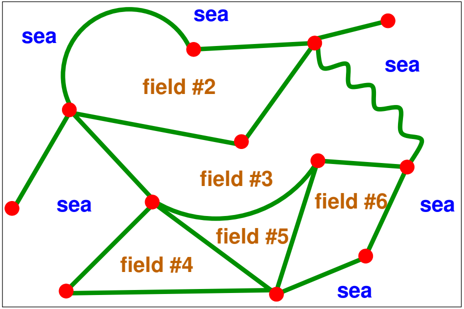
Let v, e, f denote the numbers of vertices, edges, and faces, respectively.

Then,

$$\boxed{v - e + f = 2.}$$

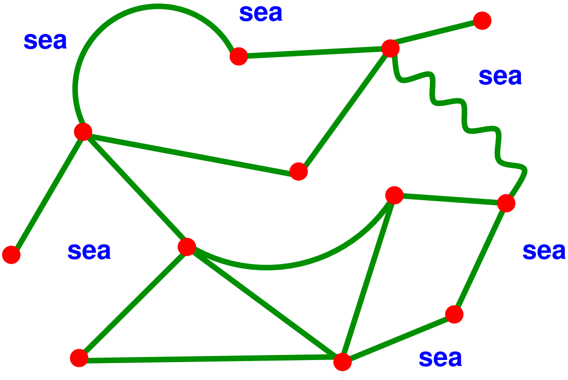
Euler's Formula: The "Raging Sea" Proof

Imagine that the edges are dikes that hold back the raging sea from a network of fields.



One by one, the dikes break under the pressure.
Each time a dike breaks, the raging sea rushes into one of the fields, and there is one fewer field than before.

Euler's Formula: The "Raging Sea" Proof



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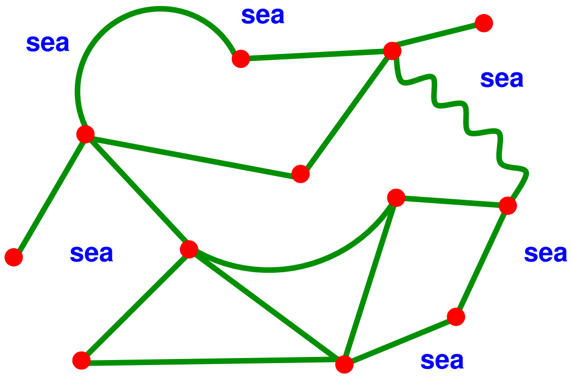
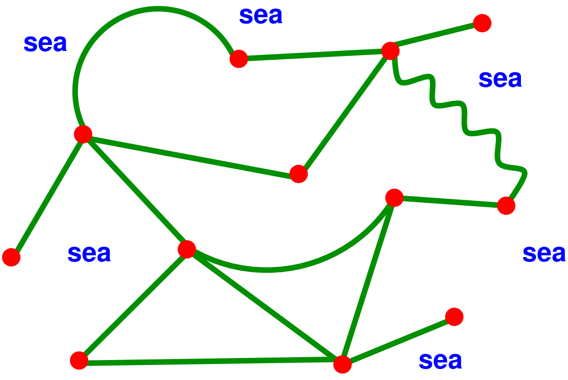


v = 12

e = 16

f = 6

Euler's Formula: The "Raging Sea" Proof

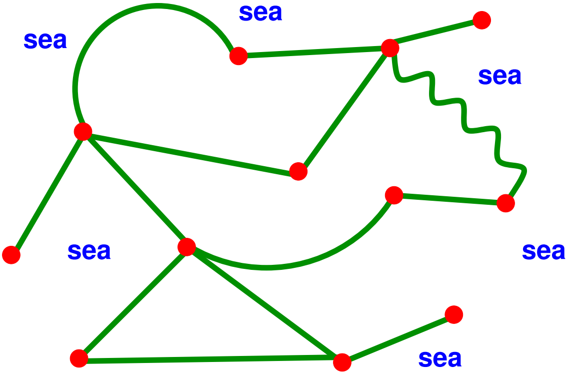


v = 12

e = 15

f = 5

Euler's Formula: The "Raging Sea" Proof

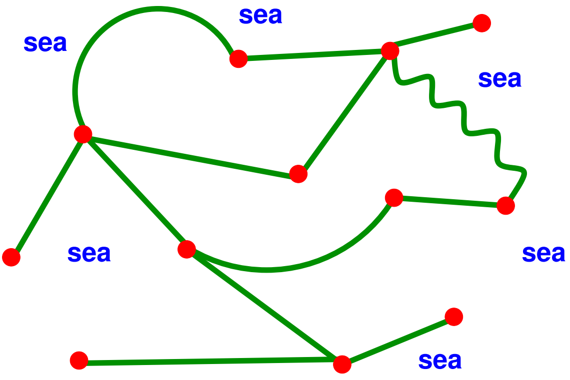


$v = 12$

$e = 14$

$f = 4$

Euler's Formula: The "Raging Sea" Proof

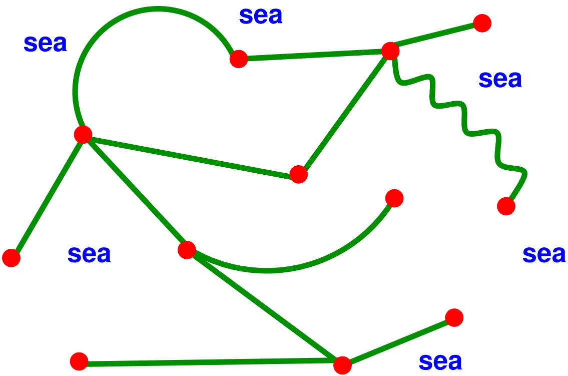


$v = 12$

$e = 13$

$f = 3$

Euler's Formula: The "Raging Sea" Proof

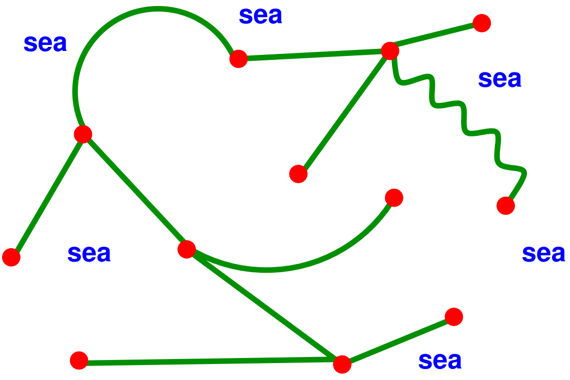


$v = 12$

$e = 12$

$f = 2$

Euler's Formula: The "Raging Sea" Proof



v = 12

e = 11

f = 1

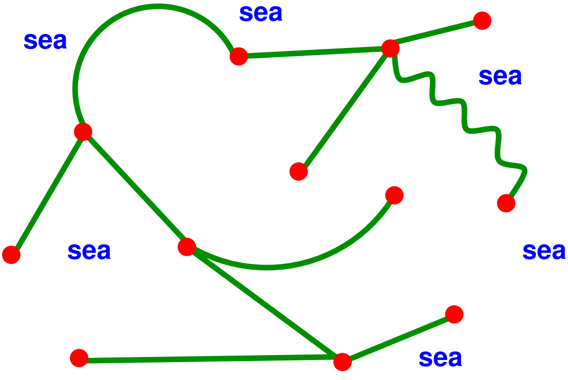
Euler's Formula: The "Raging Sea" Proof

Finally, the raging sea has overwhelmed all of the enclosed areas — the graph has only one face

There are still some dikes left, but they're now just piers extending into the sea.

One by one, the network of piers shrinks.

Euler's Formula: The "Raging Sea" Proof



v = 12
e = 11
f = 1

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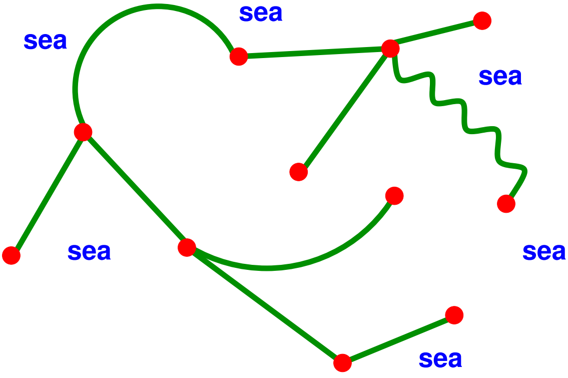

$v = 11$

$e = 10$

$f = 1$

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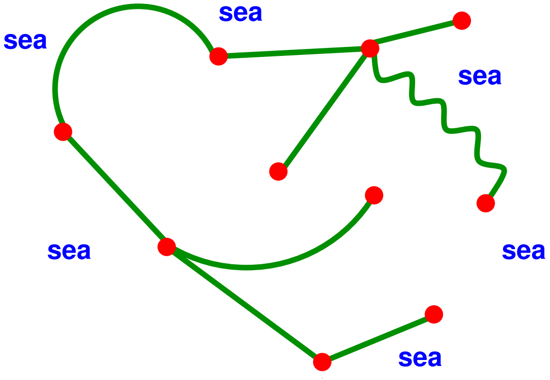

$v = 10$

$e = 9$

$f = 1$

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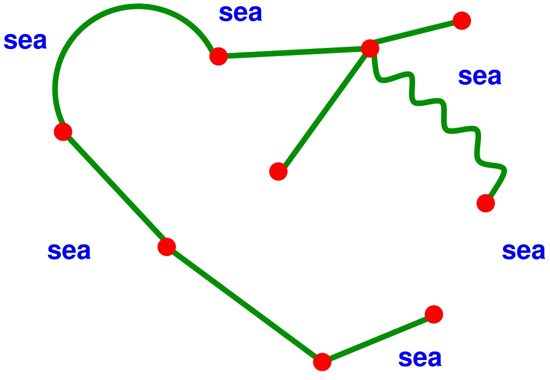
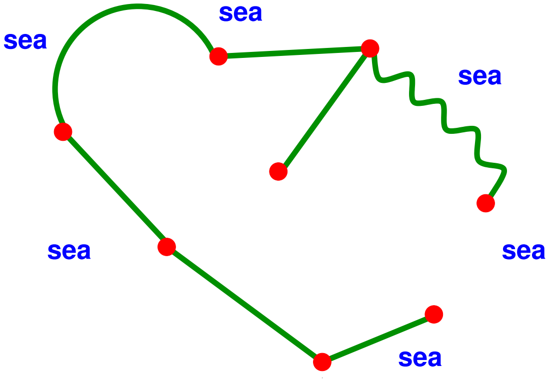


$v = 9$

$e = 8$

$f = 1$

Euler's Formula: The "Raging Sea" Proof

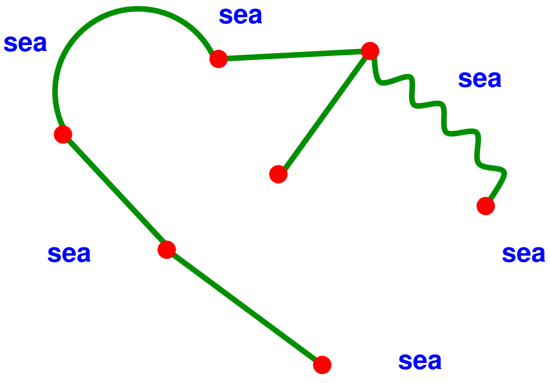


$v = 8$

$e = 7$

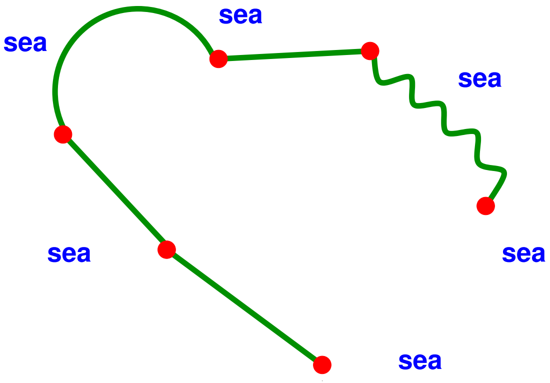
$f = 1$

Euler's Formula: The "Raging Sea" Proof



$v = 7$
$e = 6$
$f = 1$

Euler's Formula: The "Raging Sea" Proof

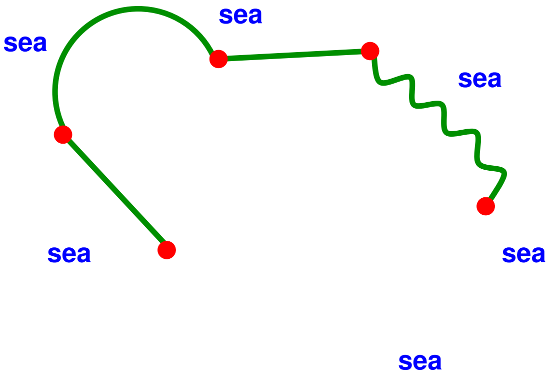


$v = 6$

$e = 5$

$f = 1$

Euler's Formula: The "Raging Sea" Proof

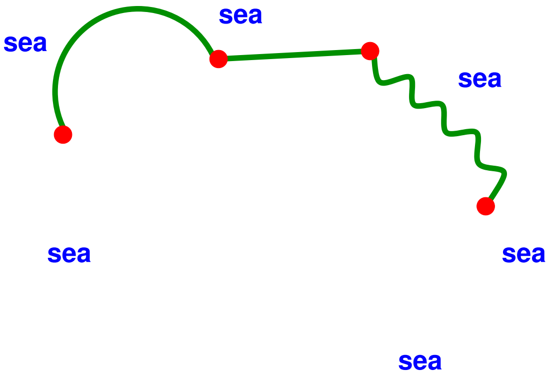


v = 5

e = 4

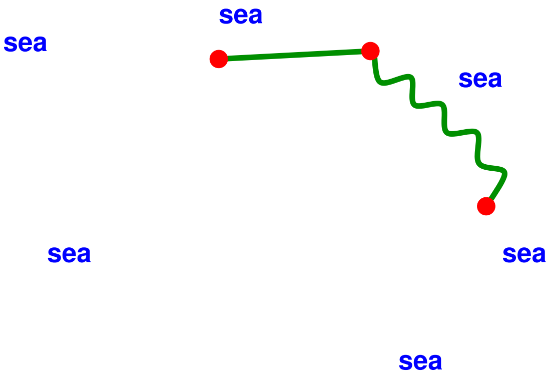
f = 1

Euler's Formula: The "Raging Sea" Proof



v = 4
e = 3
f = 1

Euler's Formula: The "Raging Sea" Proof

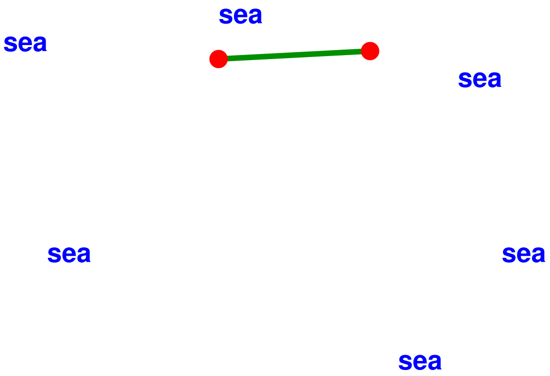


$v = 3$

$e = 2$

$f = 1$

Euler's Formula: The "Raging Sea" Proof

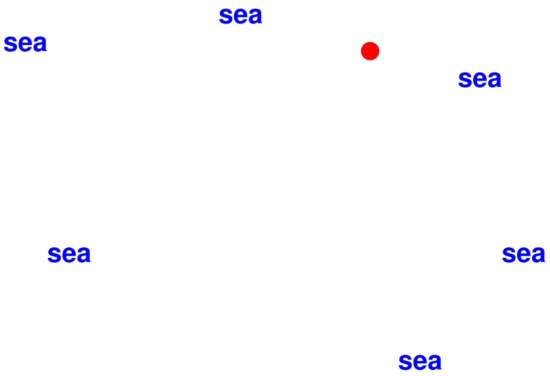


v = 2

e = 1

f = 1

Euler's Formula: The "Raging Sea" Proof



v = 1

e = 0

f = 1

Euler's Formula: The "Raging Sea" Proof

- After all the dikes (= edges) are gone,

$$v = 1, \quad e = 0, \quad f = 1, \qquad v - e + f = 2.$$

Euler's Formula: The "Raging Sea" Proof

- After all the dikes ($=$ edges) are gone,

$$v = 1, \quad e = 0, \quad f = 1, \qquad v - e + f = 2.$$

- Each time a dike disappeared, either f or v decreased by 1.

Euler's Formula: The "Raging Sea" Proof

- After all the dikes (= edges) are gone,

$$v = 1, \quad e = 0, \quad f = 1, \qquad v - e + f = 2.$$

- Each time a dike disappeared, either f or v decreased by 1.
- Therefore, the value of $v - e + f$ never changed — it must always have been 2!

Euler's Formula: The "Raging Sea" Proof

- After all the dikes (= edges) are gone,

$$v = 1, \quad e = 0, \quad f = 1, \qquad v - e + f = 2.$$

- Each time a dike disappeared, either f or v decreased by 1.
- Therefore, the value of $v - e + f$ never changed — it must always have been 2!
- This logic works no matter what the original graph was — so we have proved Euler's formula for all planar graphs.

Proofs and The Book

Most mathematicians would call this proof "beautiful" or a "proof from the Book" — it is a simple and elegant proof of an extremely important theorem.

The mathematician Paul Erdős spoke of "The Book," in which God records the best and most beautiful proofs of mathematical theorems. Once in a while, a mortal is allowed a glimpse of the Book. Erdős said:

> *"You don't have to believe in God, but you should believe in the Book."*

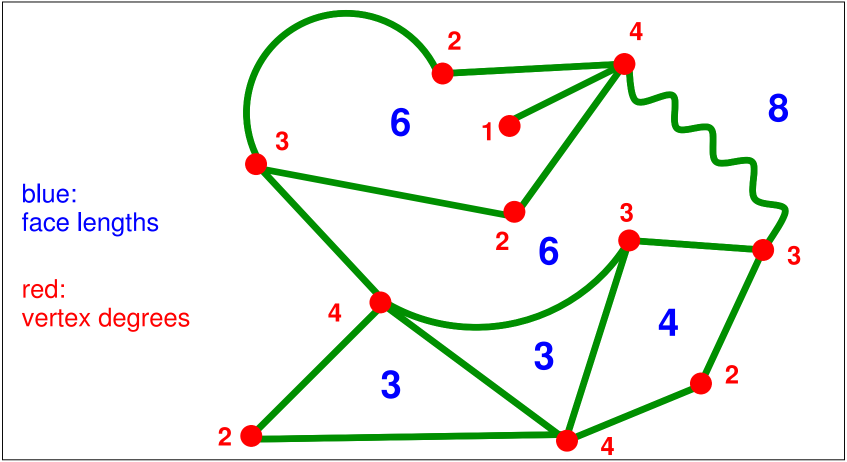
Degrees and Lengths

First, a little more terminology.

The degree $d(V)$ of a vertex v in a graph G is the number of edges having v as an endpoint.

The length $\ell(F)$ of a face F in a planar graph is the number of edges around it. (Edges that poke into F count double.)

Degrees and Lengths



Degrees, Lengths, and Euler's Formula

Theorem 1 ("Handshaking"): In a graph with e edges, the sum of the degrees of all vertices is $2e$.

(Reason: Every edge contributes to the degrees of two vertices.)

Theorem 2: In a planar graph with e edges, the sum of the lengths of all faces is $2e$.

- In a planar graph, each face has length 3 or greater, so $2e \geq 3f$, so $f \leq 2e/3$.
- By Euler's formula, $v - e + f = 2 \leq v - e + 2e/3 = v - e/3$.
- Applying a little algebra to $2 \leq v - e/3$ gives us $e \leq 3v - 6$.

The Six-Color Theorem

Theorem

Every planar graph can be colored with 6 or fewer colors.

Idea: Find the right order in which to color the vertices so that we will never need more than 6 colors.

- The trick is to think about which vertex to color last.
- It should be a vertex with at most 5 neighbors.
- How do we know G has such a vertex?

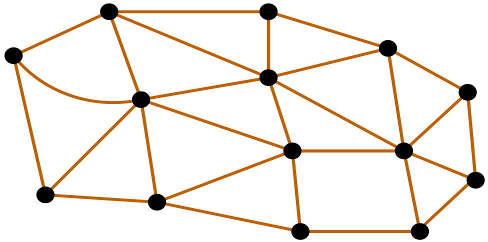
The Six-Color Theorem

Suppose G is a planar graph.

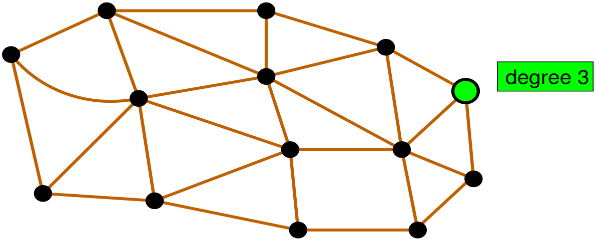
- If every vertex of G has degree 6 or greater, then the sum of degrees would be at least $6v$, and there would be at least $3v$ edges.
- But this is impossible since $e \leq 3v - 6$.
- Therefore, there must be some vertex whose degree is 5 or less. (In fact, there will be many choices — but we only need one.)

This is the vertex we are going to color last.

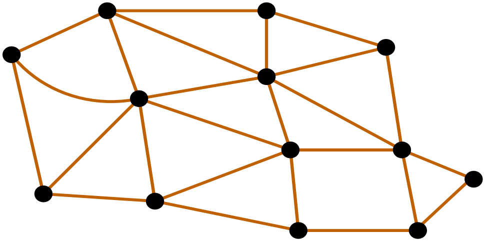
Successively Deleting Vertices of Degree 6 Or Less



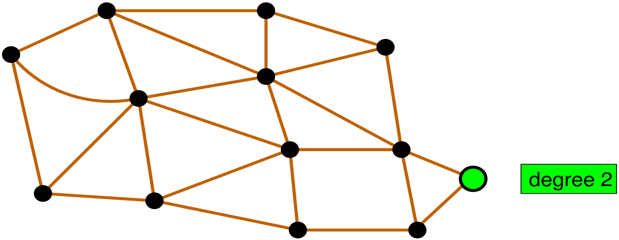
Successively Deleting Vertices of Degree 6 Or Less



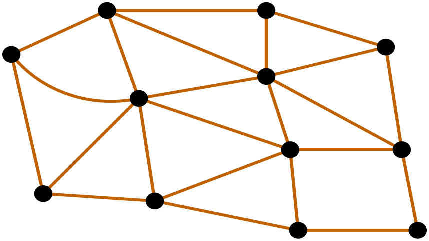
Successively Deleting Vertices of Degree 6 Or Less



Successively Deleting Vertices of Degree 6 Or Less



Successively Deleting Vertices of Degree 6 Or Less



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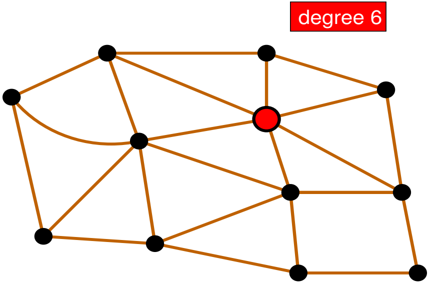

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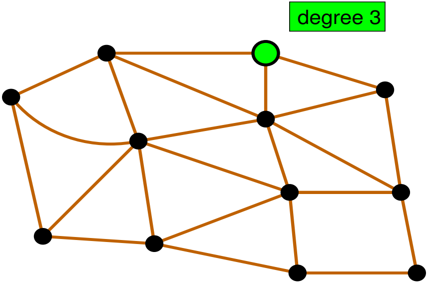

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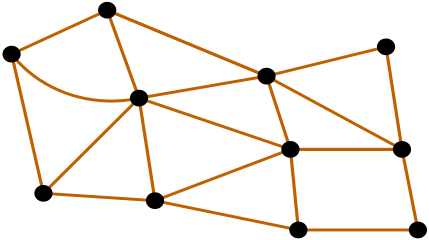
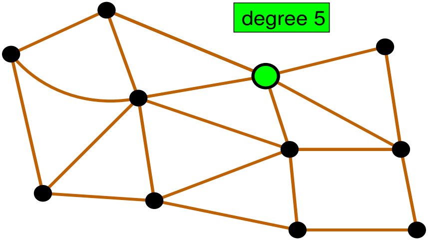


Successively Deleting Vertices of Degree 6 Or Less



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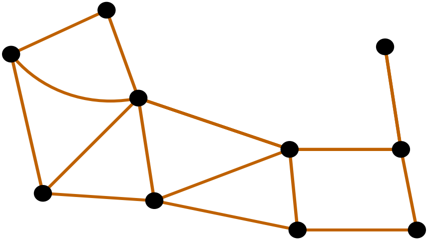
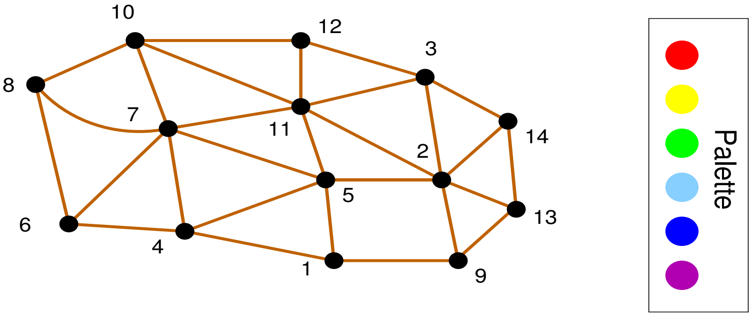


Successively Deleting Vertices of Degree 6 Or Less

Et cetera.

Color the vertices in reverse order — last deleted = first colored.

Always use the first possible color from the palette.



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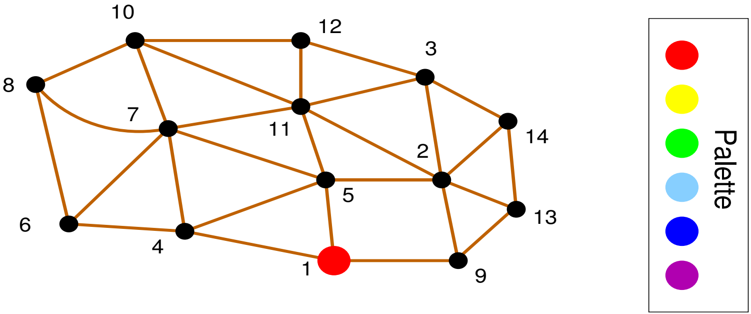

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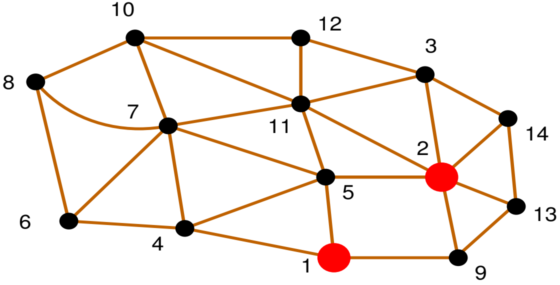

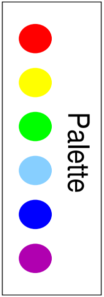

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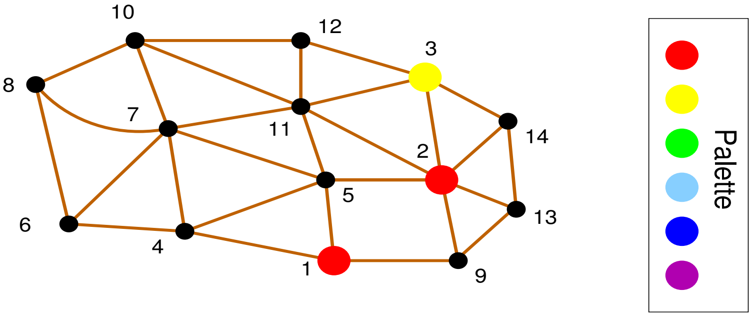

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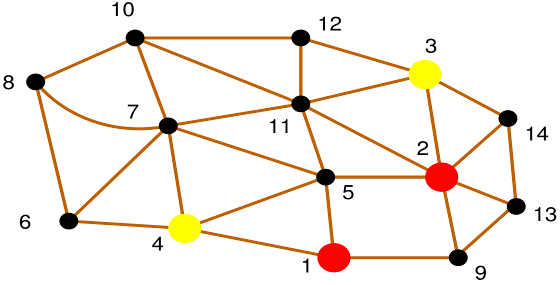

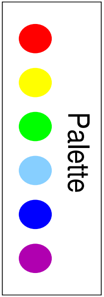

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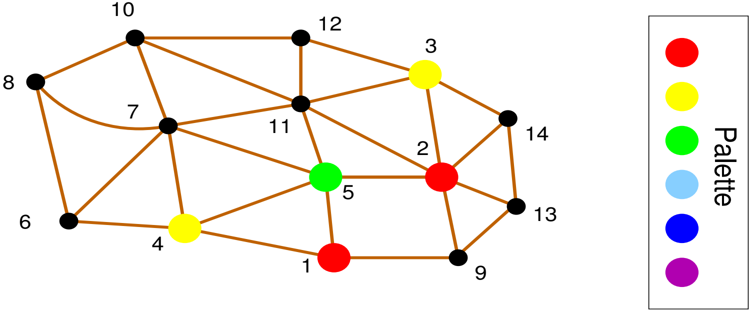

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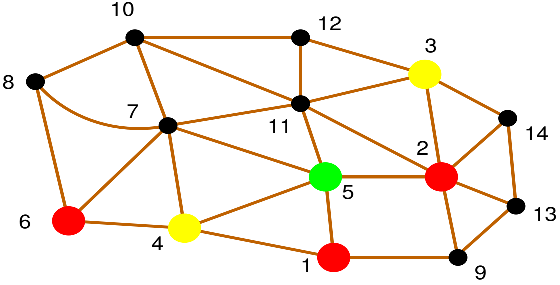

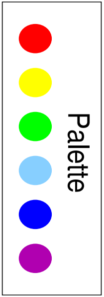

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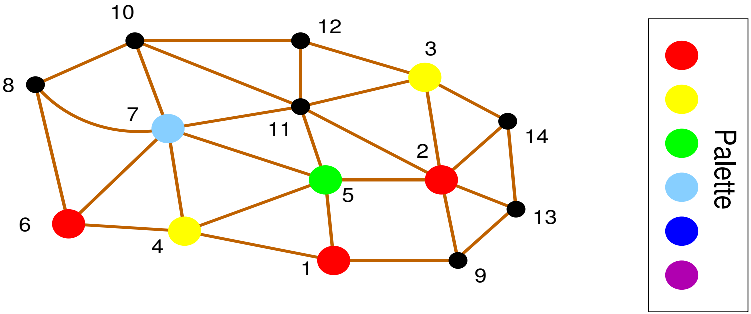

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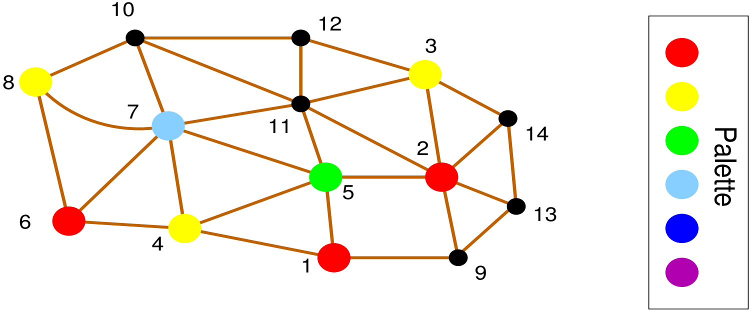

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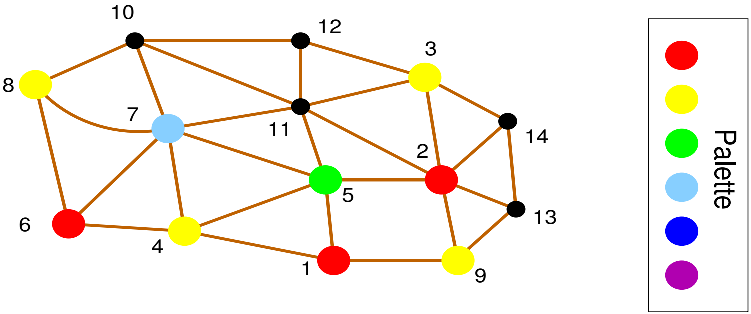

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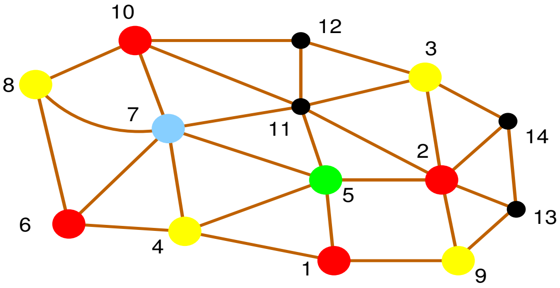

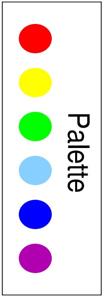

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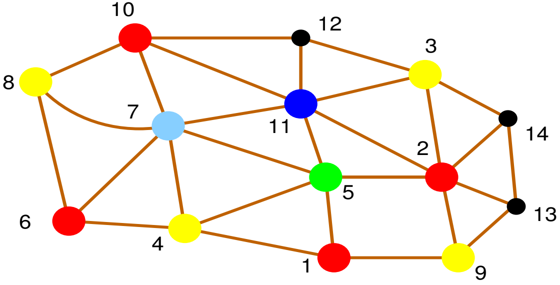

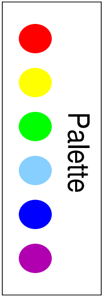

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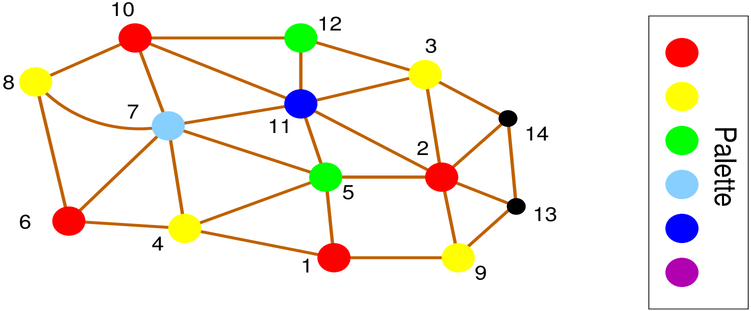

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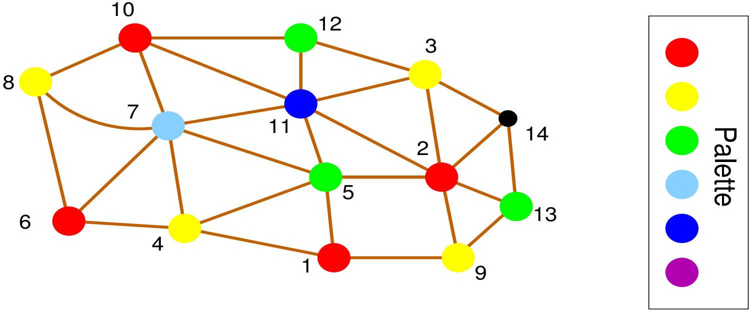

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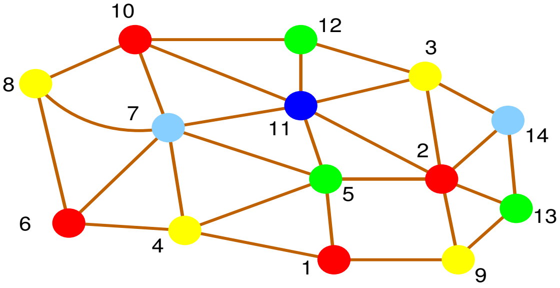

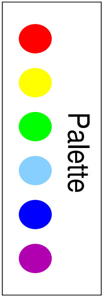

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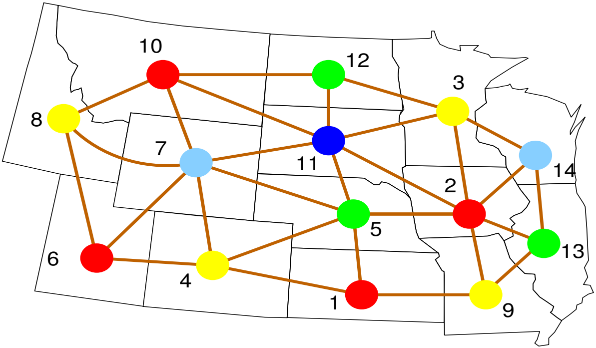

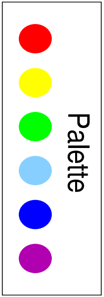

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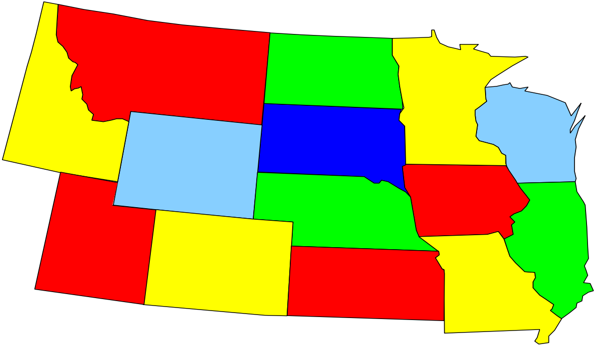

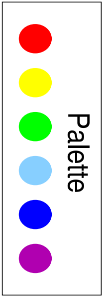

Proof of the 6CT

- Each time we color a new vertex, it has at most 5 neighbors that have already been colored.

With a little work, the method can be souped up to prove that every planar graph is 5-colorable.

(Idea: If we happen to require color #6, show that there is some way of going back and modifying the previous coloring so that color #6 is not really required.)

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Proof of the 6CT

- Each time we color a new vertex, it has at most 5 neighbors that have already been colored.
- Therefore, at least one color from the palette is available for every vertex.
- This method works for any planar graph — so every planar graph is 6-colorable.

With a little work, the method can be souped up to prove that every planar graph is 5-colorable.

(Idea: If we happen to require color #6, show that there is some way of going back and modifying the previous coloring so that color #6 is not really required.)

Interlude: What Is A Proof?

- What is a mathematical proof?
  - A logical argument that relies on commonly accepted axioms and rules of inference (e.g., "if $a = b$ and $b = c$, then $a = c$").
- When is a proof correct?
  - The standard of proof is very high in mathematics (not just "beyond a reasonable doubt", but beyond **any** doubt)
- Who gets to decide whether a proof is correct?
- Are some proofs better than others?

Appel and Haken's Proof

In 1976, Kenneth Appel and Wolfgang Haken, of the University of Illinois, announced a proof of the Four-Color Theorem.

Suppose there exists a planar graph that requires more than 4 colors — the idea is to show that something impossible must then happen.

Observation: If there is at least one such graph, then there is a *smallest* such graph (i.e., with the minimum number of vertices).

Let G be a non-4-colorable planar graph that is as small as possible (a "minimal criminal").

Appel and Haken's Proof (More Or Less)

Suppose G is a minimal criminal.

Step 1: Prove that G contains at least one of 1476 unavoidable configurations.

(To do this, assign each vertex a "charge". Let the electrons flow around G (according to 487 "discharging rules"). If a vertex still has electrons that it cannot discharge, the reason must be that there is one of those 1476 configurations nearby.)

Step 2: Prove that each one of those 1476 unavoidable configurations is reducible — it can be replaced with something smaller without affecting the chromatic number of G.
This part of the proof was carried out by a computer.

Appel and Haken's Proof (More Or Less)

Suppose G is a minimal criminal.

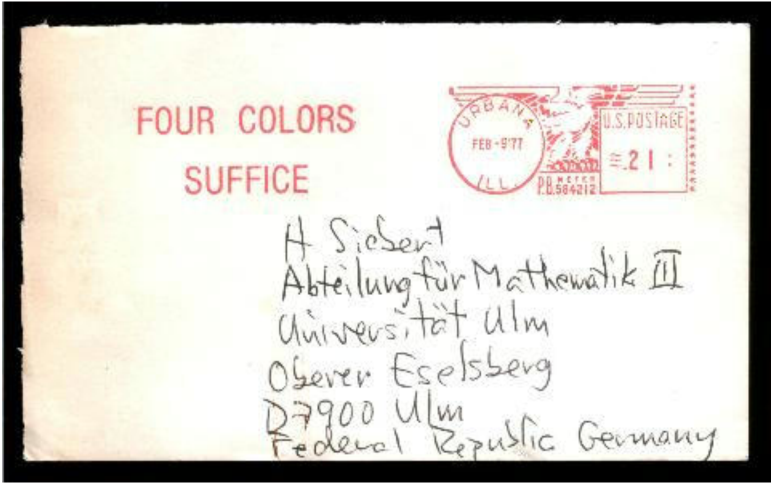
Step 1: Prove that G contains at least one of 1476 unavoidable configurations.

Step 2: Prove that each one of those 1476 unavoidable configurations is reducible.

Conclusion: G was not a minimal criminal.

There is no such thing as a minimal non-4-colorable planar graph. Therefore, there are no non-4-colorable planar graphs!

Appel and Haken's Proof



Appel and Haken's Proof

Credit where credit is due:

- Discharging: introduced by Heinrich Heesch in 1969
- Much of the code written together with graduate student John Koch
- Other experts: Frank Allaire, Jean Mayer, Ted Swart, ...
- Appel and Haken enlisted their kids to help check the list of configurations
- Ulrich Schmidt verified about 40% of the discharging part of the proof in his 1981 master's thesis (and found several small but fixable errors)

Appel and Haken's Proof

Appel and Haken, describing their proof of the 4CT:

> *This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in the 24 lemmas in the main sections of text. In addition, the reader is told that certain facts have been verified with the use of about twelve hundred hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail.*

Reactions to Appel and Haken's Proof

Philosopher Thomas Tymoczko (1979):

> *4CT is not really a theorem... [N]o mathematician has seen a proof of the 4CT, nor has any seen a proof that it has a proof. Moreover, it is very unlikely that any mathematician will ever see a proof of the 4CT.*

Mathematician and master expositor Paul Halmos (1990):

> *I do not find it easy to say what we learned from all that.*

Mathematician Daniel Cohen (1991):

> *The mission of mathematics is understanding... Admitting the computer shenanigans of A&H to the ranks of mathematics would only leave us intellectually unfulfilled.*

Reactions to Appel and Haken's Proof

Kenneth Appel:

> *The computer was . . . much more successful, because it was not thinking like a mathematician.*

Mathematician Doron Zeilberger (2002):

> *. . . Ken Appel and Wolfgang Haken's GORGEOUS proof of the Four-Color Theorem. Few people are aware that it is really a ONE-LINE Proof: 'The following finite set of reducible configurations, let's call it S, is unavoidable'. The set S itself does not have to be actually examined by human eyes, and perhaps should not. The computer would be much more reliable than any human in checking its claim. . . . FOUR COLORS SUFFICE BECAUSE THE COMPUTER SAID SO!*

Improving Appel and Haken's Proof

Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas [RSST] (1995–1997) gave an improved proof, using the same approach as Appel and Haken, but with ~~1476~~ **633** unavoidable configurations and ~~487~~ **32** discharging rules.

- The RSST proof gives a faster algorithm to explicitly compute a 4-coloring of a planar graph
- The computer-aided part of the RSST proof is available online (you can download the code and documentation and run it for yourself)

Formal Proof

Instead of using a computer to check hundreds or thousands of cases, why not use a computer to check the proof itself?

- Teach a computer how to check logical proofs (free software exists for this)
- Translate a proof into language understandable by the computer (needs to be very precise!)
- In order to believe the theorem, you don't have to check every line of the proof — you only have to trust that the proof-checker itself was programmed correctly.

2005: Formal proof for 4CT published by computer scientist Georges Gonthier

Formal Proof

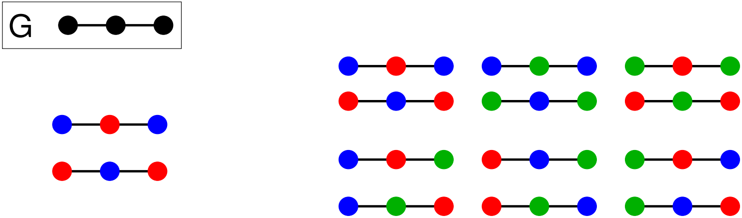
Gonthier:

> *The approach that proved successful for this proof was to turn almost every mathematical concept into a data structure or a program, thereby converting the entire enterprise into one of program verification.*

> *Perhaps this is the most promising aspect of formal proof: it is not merely a method to make absolutely sure we have not made a mistake in a proof, but also a tool that shows us and compels us to understand why a proof works.*

Chromatic Polynomials of Graphs

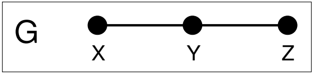
Suppose G is a graph and k is a number.
How many proper colorings of G are there using k or fewer colors?
Call this number $C(G, k)$.



C(G,2) = 2

C(G,3) = 12

Chromatic Polynomials of Graphs

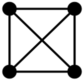
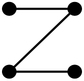
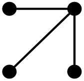
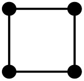


What is $C(G,k)$ for this graph?

- Vertex X: k available colors
- Vertex Y: $k-1$ available colors (can't use color of X)
- Vertex Z: $k-1$ available colors (can't use color of Y)

Conclusion: $C(G,k) = k(k-1)^2$.

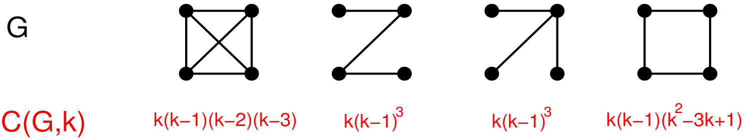
Chromatic Polynomials of Graphs

G				
C(G,k)	k(k–1)(k–2)(k–3)	$k(k-1)^3$	$k(k-1)^3$	$k(k-1)(k^2-3k+1)$

Remarkable Fact $C(G,k)$ is always a polynomial function of k (called the chromatic polynomial).

Idea (Birkhoff, Whitney, early 20th century): Maybe we can use algebra to prove that if G is planar, then $C(G,4) \neq 0$.

Chromatic Polynomials of Graphs



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No one has figured out how to make this work.

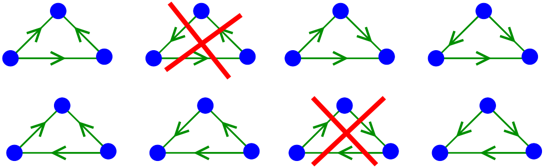
Chromatic Polynomials of Graphs

Even though the chromatic polynomial did not help solve the Four-Color Problem, it has turned out to be of great theoretical interest!

- When does it factor into linear terms?
- What do its coefficients tell you about the graph?
- What other information about G can you obtain from $C(G, k)$?
- Which polynomials actually are chromatic polynomials?
- What other polynomials encode structure of G? How are they related to the chromatic polynomial?

Stanley's Amazing Theorem

Draw an arrow along each edge of G. Make sure that there are no closed circuits. This is called an acyclic orientation (or AO)



Here $C(G,k) = k(k-1)(k-2)$ and there are 6 AOs.

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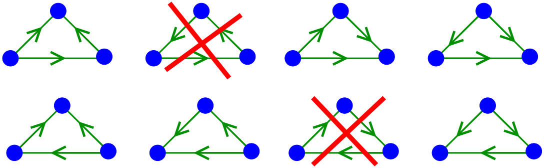


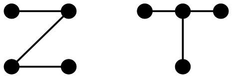
Here $C(G,k) = k(k-1)(k-2)$ and there are 6 AOs.

Theorem (Richard Stanley, 1973)

For every graph G, the number of AOs of G is $|C(G,-1)|$.

A More Detailed Census of Colorings

Every tree with n vertices has chromatic polynomial $k(k-1)^{n-1}$.



$P(k) = k(k–1)^3$

$P(4) = 4 \times 3^3 = 108$

- The chromatic polynomial itself cannot distinguish Z and T.
- In order to tell Z and T apart, we need a more detailed census of colorings.
- Idea: Count colorings that use each possible "palette" (e.g., ●●●● or ●●●●).

The Chromatic Symmetric Function

Palette	Colorings of Z	Colorings of T
●●●●	0	0
●●●●	0	1
●●●●	2	0
●●●●	6	6
●●●●	24	24

This list of data (number of colorings for each possible palette) is called the chromatic symmetric function of a graph, introduced by Stanley in 1995.

Question: Can two different trees have the same chromatic symmetric function?

The Chromatic Symmetric Function

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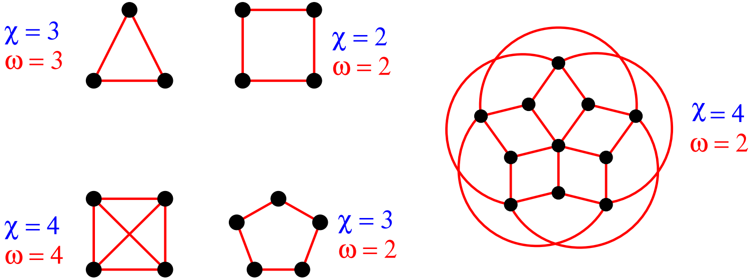
- This is an open problem (and seems very hard!)
- Some theoretical results about information contained in the chromatic symmetric function (Martin–Morin–Wagner, 2006)
- Some special kinds of trees can be identified from their chromatic symmetric functions (Aliste–Zamora, 2012)
- If two such trees exist, they must have 26 or more vertices (Keeler Russell, KU senior project, 2013)

Perfect Graphs

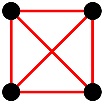
The chromatic number χ is the fewest colors needed to color G properly. The clique number ω is the greatest number of mutually adjacent vertices in G.



In general $\chi \geq \omega$, but the two numbers **need not be equal.**

Perfect Graphs

A graph G is perfect if $\chi = \omega$ **for any induced subgraph.** (What's an induced subgraph? Choose some of the vertices of G and all the edges that connect the vertices you've chosen.)



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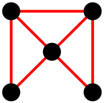

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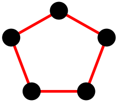
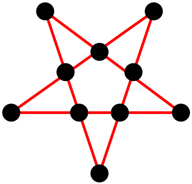


Not perfect:
$\chi = 3$
$\omega = 2$



Not perfect, even though $\chi = 3$ and $\omega = 3$.

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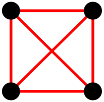

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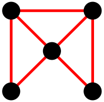

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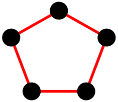

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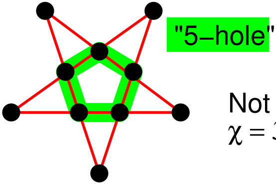

Not perfect, even though $\chi = 3$ and $\omega = 3$.

Perfect Graphs

- Coloring perfect graphs is much easier than coloring arbitrary graphs.
- Many other algorithms work better if the graph is known to be perfect.
- Many interesting classes of graphs are perfect.

Theorem (Lovász 1972)

G is perfect if and only if its complement $\bar{G}$ is perfect.
(I.e., replace all edges by non-edges and all non-edges by edges.)

Theorem (Chudnovsky, Robertson, Seymour, Thomas 2006)

G is perfect if and only if neither G nor $\bar{G}$ has an odd hole.

Further Reading: Articles and Books

- K. Appel and W. Haken, "The Solution of the Four-Color Map Problem," *Scientific American*, vol. 237 (1977), 108–121
- D. Barnette, *Map Coloring, Polyhedra, and the Four-Color Problem*, Mathematical Association of America, 1983
- D. Richeson, *Euler's Gem*, Princeton University Press, 2008
- R. Thomas, "An Update on the Four-Color Theorem," *Notices of the American Mathematical Society*, vol. 45, no. 7 (August 1998), 848–859
- R. Wilson, *Four Colours Suffice*, Penguin, 2002

Further Reading: Links

- Wikipedia page on graph theory:
  en.wikipedia.org/wiki/Graph_theory
- Wikipedia page on the 4CT:
  en.wikipedia.org/wiki/Four_color_theorem
- 4CT at MacTutor History of Mathematics Archive:
  www-history.mcs.st-andrews.ac.uk/HistTopics/
  The_four_colour_theorem.html
- Robin Thomas's 4CT page (including downloadable code and documentation):
  people.math.gatech.edu/∼thomas/FC/fourcolor.html