The Support of Kostant's Weight Multiplicity Formula is an Order Ideal in the Weak Bruhat Order

Portia X. Anderson, Esther Banaian, Melanie J. Ferreri, Owen C. Goff, Kimberly P. Hadaway, Pamela E. Harris, Kimberly J. Harry, Nicholas Mayers, Shiyun Wang, Alexander Wilson*

Overview

1. What (and Why) is a Weyl Alternation Set?

2. Poset Structure

3. Proof Sketch

What (and Why) is a Weyl Alternation Set?

Lie Algebras

A **Lie algebra** $\mathfrak{g}$ is a vector space over $\mathbb{C}$ equipped with an operation called a Lie bracket.

$\mathfrak{sl}_n$

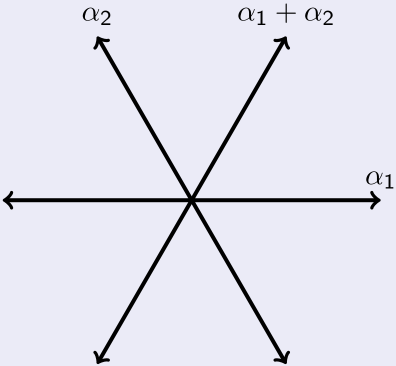
The Lie algebra $\mathfrak{sl}_n$ consists of $(n+1) \times (n+1)$ matrices over $\mathbb{C}$ with trace zero and Lie bracket

$$[X, Y] = XY - YX.$$

Roots

A simple Lie algebra $\mathfrak{g}$ has an associated irreducible **root system** Φ, which we'll think of as vectors in Euclidean space.

$\mathfrak{sl}_2$



- Simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$
- Positive Roots Φ^+
- Negative Roots $\Phi^- = -\Phi^+$

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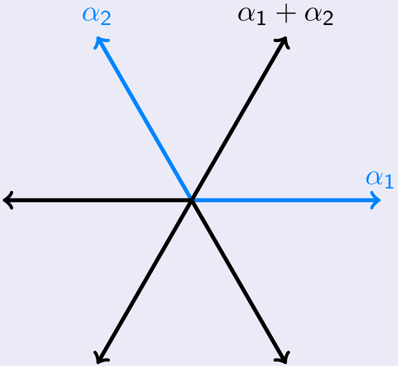

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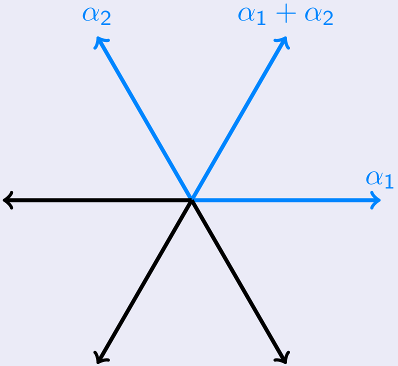

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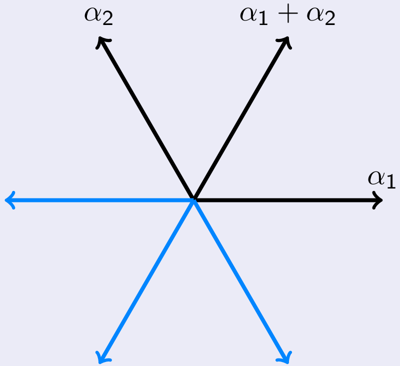

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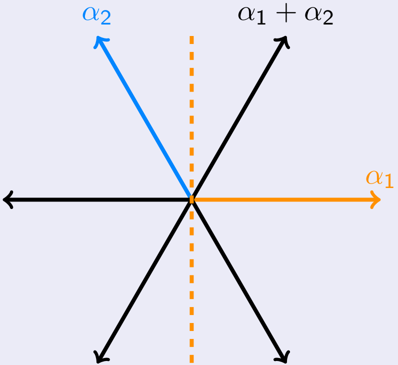
Weyl Group

For a root system with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, the corresponding **Weyl group** W is generated by reflections $s_1, \ldots, s_r$ where s_i is the reflection through the hyperplane orthogonal to α_i.

$\mathfrak{sl}_2$



$$s_1(\alpha_2) = \alpha_1 + \alpha_2$$

Weights

- A **representation** V of $\mathfrak{g}$ is a map $\mathfrak{g} \to \mathfrak{gl}(V)$ respecting the Lie bracket.
- A **weight space** is a generalized eigenspace. Formally, if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then a **weight** λ is a linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$, and the corresponding weight space is
$$V_\lambda = \{v \in V | \forall H \in \mathfrak{h}, Hv = \lambda(H)v\}.$$
- A simple $\mathfrak{g}$-representation is determined by its highest weight. For V the representation with highest weight λ, we we write
$$m(\lambda, \mu) = \dim(V_\mu)$$
for the **multiplicity** of μ in V.

Weights

We can think of weights as living in Euclidean space along with roots *(roots are weights of the adjoint representation)*.

- Write (λ, α) for the inner product in this Euclidean space
- The Weyl group acts as $s_i(\lambda) = \lambda - 2\frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$

Property	**Definition**
λ dominant	$(\lambda, \alpha) \geq 0$ for all $\alpha \in \Phi^+$
λ integral	$2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$
$\lambda \leq \mu$	$\mu - \lambda$ can be written as a positive linear combination of positive roots

We say that μ is **higher** than λ whenever $\lambda < \mu$.

Kostant's Weight Multiplicity Formula

Theorem (Kostant 1958)

The multiplicity of the weight μ in the representation V of $\mathfrak{g}$ with highest weight λ is

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - \mu - \rho)$$

where

- $\ell(\sigma)$ *is the minimum number of reflections needed to write* σ,
- $\wp(\xi)$, *the* **Kostant partition function**, *is the number of ways to write* ξ *as a non-negative integer linear combination of positive roots* Φ^+, *and*
- $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Kostant's Partition Function

Example

$\wp(2\alpha_1 + 3\alpha_2) = 3$

1. $2(\alpha_1) + 3(\alpha_2) + 0(\alpha_1 + \alpha_2)$
2. $1(\alpha_1) + 2(\alpha_2) + 1(\alpha_1 + \alpha_2)$
3. $0(\alpha_1) + 1(\alpha_2) + 2(\alpha_1 + \alpha_2)$

Example

$\wp(3\alpha_1 - \alpha_2) = 0$

Note: $\wp(\alpha) = 0$ if and only if α has a negative coefficient when expanded as a linear combination of simple roots.

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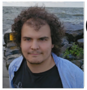

Owen Goff
★★★★★

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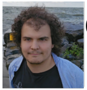

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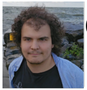

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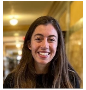


Esther Banaian
★☆☆☆☆

Tried to use it for the type B8 Lie algebra, but it told me I had to sum over more than 10 million terms! What gives??

Weyl Alternation Set

For some elements $\sigma \in W$, we have that $\wp(\sigma(\lambda+\rho)-\mu-\rho)=0$, so they don't contribute to the sum. The **Weyl alternation set** is the set of elements that *do* contribute:

$$\mathcal{A}(\lambda,\mu)=\{\sigma \in W : \wp(\sigma(\lambda+\rho)-\mu-\rho)>0\}$$

Note that $\sigma \in \mathcal{A}(\lambda,\mu)$ if and only if $\sigma(\lambda+\rho)-\mu-\rho$ is a linear combination of positive roots with nonnegative (not all zero) coefficients.

We can take the sum in our formula over only elements of $\mathcal{A}(\lambda,\mu)$ instead of the full Weyl group.

Poset Structure

Weak Order

A **reduced expression** of an element $\sigma \in W$ is a minimum length expression for σ as a product of simple transpositions s_i.

The **left weak order** $(W, \leq_L)$ is defined by $\sigma \leq_L \tau$ if a reduced expression for σ is a suffix of a reduced expression for τ.

Example

$$s_1 s_3 \leq_L s_1 s_2 s_1 s_3$$

The **right weak order** $(W, \leq_R)$ is defined by $\sigma \leq_R \tau$ if a reduced expression for σ is a prefix of a reduced expression for τ.

Example

$$s_1 s_2 \leq_R s_1 s_2 s_1 s_3$$

Poset Structure of $\mathcal{A}(\lambda, \mu)$

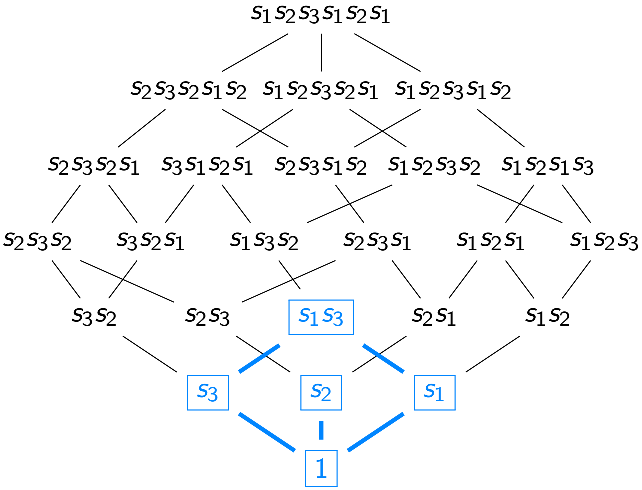
Theorem

Let λ be an integral dominant weight of a simple Lie algebra $\mathfrak{g}$ with Weyl group W. Then for any weight μ, the Weyl alternation set $\mathcal{A}(\lambda, \mu)$ is a (possibly empty) order ideal in the left and right weak orders of W.

Corollary

If $\sigma \in \mathcal{A}(\lambda, \mu)$, then any contiguous subword of a reduced expression for σ is also in $\mathcal{A}(\lambda, \mu)$.

Poset Structure of $\mathcal{A}(\lambda, \mu)$



The left weak order on the type A_3 Weyl group with the set

$$\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$$

highlighted where

$$\tilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3.$$

Forbidden Words

Corollary

If $\sigma \in \mathcal{A}(\lambda, \mu)$, then any contiguous subword of a reduced expression for σ is also in $\mathcal{A}(\lambda, \mu)$.

Kimberly P. Hadaway

Here's a clever hack: the contrapositive of this theorem doubles as a way to prove elements *aren't* in the alternation set!

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Contrapositive

If $\sigma \notin \mathcal{A}(\lambda, \mu)$, then any word containing a reduced expression of σ as a contiguous subword is also not in $\mathcal{A}(\lambda, \mu)$.

Proof Sketch

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Suppose that σ is covered by τ in the right weak order. This means that

$$\tau = \sigma s_i, \text{ and}$$
$$\ell(\sigma s_i) > \ell(\sigma)$$

The latter statement implies that $\sigma(\alpha_i) \in \Phi^+$. Some routine computations reveal that

$$\sigma(\lambda + \rho) - \mu - \rho = \tau(\lambda + \rho) - \mu - \rho + 2\frac{(\lambda + \rho, \alpha_i)}{(\alpha_i, \alpha_i)}\sigma(\alpha_i).$$

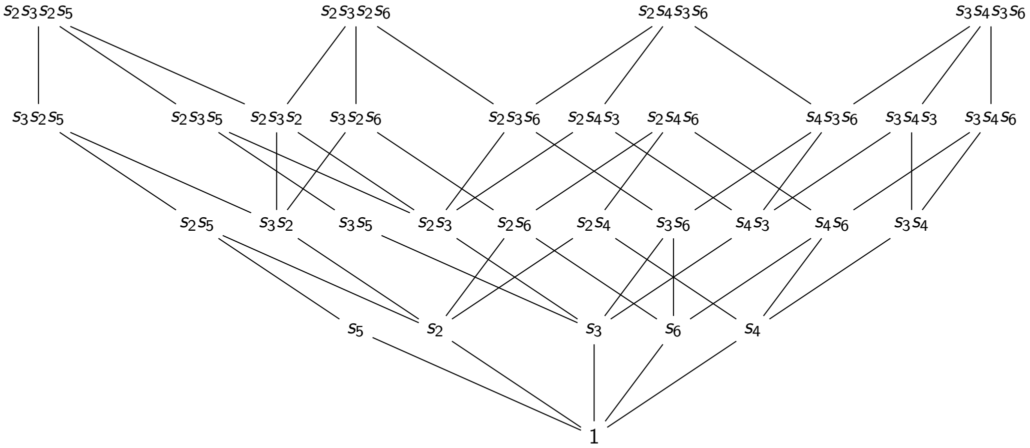
Because λ and ρ are dominant integral weights, $2\frac{(\lambda+\rho,\alpha_i)}{(\alpha_i,\alpha_i)}$ is a non-negative integer.

Proof Sketch

$$\sigma(\lambda+\rho)-\mu-\rho=\tau(\lambda+\rho)-\mu-\rho+2\frac{(\lambda+\rho,\alpha_i)}{(\alpha_i,\alpha_i)}\sigma(\alpha_i)$$

Observe that if $\tau(\lambda+\rho)-\mu-\rho$ is a non-negative-integral linear combination of simple roots (i.e. $\tau\in\mathcal{A}(\lambda,\mu)$), then the result of adding a non-negative multiple of a positive root will also be such a non-negative-integral linear combination of simple roots (i.e. $\sigma\in\mathcal{A}(\lambda,\mu)$).

A Pretty Picture of $\mathcal{A}_7(\tilde{\alpha}, -\alpha_2 - \alpha_3 - \alpha_4)$



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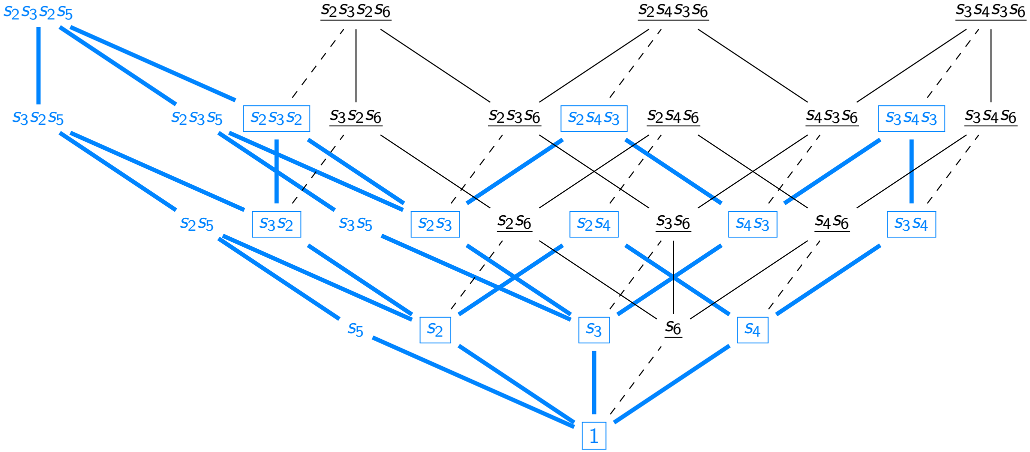

- The blue subposet with thick edges is $\mathcal{A}_6(\tilde{\alpha}, \mu)$
- The subposet of boxed values is $\mathcal{A}_5(\tilde{\alpha}, \mu)$

Summary

- The Weyl alternation sets $\mathcal{A}(\lambda, \mu)$ cut down on the computations necessary to compute the multiplicity $m(\lambda, \mu)$ of the weight μ in the irreducible representation with highest weight λ.
- For λ a dominant integral weight, $\mathcal{A}(\lambda, \mu)$ is an order ideal simultaneously in the left and right weak Bruhat orders.
- If you want to hear more about building up these Weyl alternation sets recursively, stick around for Kimberly's talk.

Thank you!

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Contact email: math@alexandernwilson.com

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