# Scarf complex of powers of an extremal ideal

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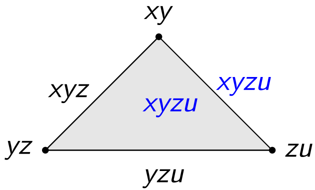
# Free Resolutions

### Example

Let $I = (xy, yz, zu)$, a monomial ideal in $S = k[x, y, z, u]$.
A free resolution of $S/I$ over $S$:

$$0 \longrightarrow \underset{\beta_3=1}{S(xyzu)} \xrightarrow{\begin{bmatrix} u \\ x \\ -1 \end{bmatrix}} \underset{\beta_2=3}{\begin{matrix} S(xyz) \\ \oplus \\ S(yzu) \\ \oplus \\ S(xyzu) \end{matrix}} \xrightarrow{\begin{bmatrix} z & 0 & zu \\ -x & u & 0 \\ 0 & -y & -xy \end{bmatrix}} \underset{\beta_1=3}{\begin{matrix} S(xy) \\ \oplus \\ S(yz) \\ \oplus \\ S(zu) \end{matrix}} \longrightarrow S$$

# Taylor complex [Taylor '66]

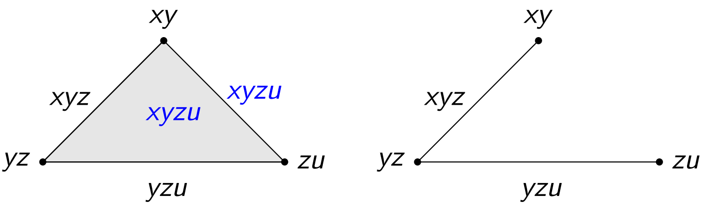



- vertices = monomial generators
- faces = lcm
- always a simplex

The Taylor complex supports the free resolution of any monomial ideal (but not minimally)

$$0 \longrightarrow S(xyzu) \xrightarrow{\begin{bmatrix} u \\ x \\ -1 \end{bmatrix}} \begin{matrix} S(xyz) \\ \oplus \\ S(yzu) \\ \oplus \\ S(xyzu) \end{matrix} \xrightarrow{\begin{bmatrix} z & 0 & zu \\ -x & u & 0 \\ 0 & -y & -xy \end{bmatrix}} \begin{matrix} S(xy) \\ \oplus \\ S(yz) \\ \oplus \\ S(zu) \end{matrix} \longrightarrow S$$

# Scarf complex [BPS '98]




Scarf complex removes all matching faces with same label.

- It might not support a free resolution ...
- ... but it is a lower bound:
- Up to isomorphisms, any free resolution of monomial ideal $I$ contains $\mathrm{Scarf}(I)$ as a subcomplex.

$$0 \longrightarrow \underset{\beta_2=2}{S(xyz)\oplus S(yzu)} \xrightarrow{\begin{bmatrix} z & 0 \\ -x & u \\ 0 & -y \end{bmatrix}} \underset{\beta_1=3}{S(xy)\oplus S(yz)\oplus S(zu)} \longrightarrow S$$

# Powers

Power of a (monomial) ideal:

$$I^2 = (m_1, \ldots, m_p)^2 = (\{m_i m_j : 1 \leq i \leq j \leq p\}).$$

Similarly for $I^r$ when $r > 2$.
Taylor is never minimal for $I^r$ when $r \geq 2$.

# Extremal ideal [CEFMMŞS '24]

$$S_{\mathcal{E}} = k[x_A \colon \emptyset \neq A \subseteq [q]]$$

$$\epsilon_i = \prod_{i \in A \subseteq [q]} x_A$$

Then $\mathcal{E}_q = (\epsilon_1, \ldots, \epsilon_q)$ is the extremal ideal

$$\mathcal{E}_3 = (\epsilon_1 = x_1 x_{12} x_{13} x_{123}, \epsilon_2 = x_2 x_{12} x_{23} x_{123}, \epsilon_3 = x_3 x_{13} x_{23} x_{123})$$

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$$\beta_i(I^r) \leq \beta_i(\mathcal{E}_q^r)$$

where $I$ is any ideal generated by $q$ square-free monomials. So we focus on $\mathbb{S}_q^r = \mathrm{Scarf}(\mathcal{E}_q^r)$.

- What does it look like?
- Does it always support a minimal free resolution of $\mathcal{E}_q^r$?

# An integer programming description of $\mathbb{S}_q^r$

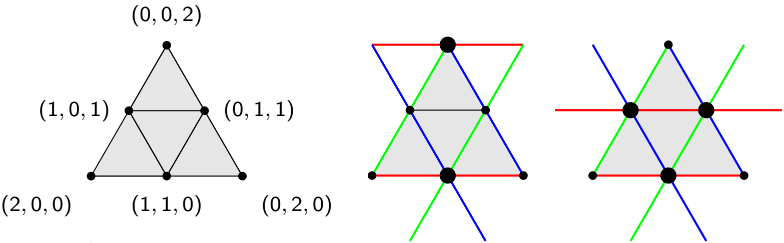
**Theorem (EFŞS '24)**

*Let* $\sigma = \{\boldsymbol{\epsilon}^{\mathbf{a}_1}, \ldots, \boldsymbol{\epsilon}^{\mathbf{a}_d}\} \in \mathbb{T}_q^r = \mathit{Taylor}(\mathcal{E}_q^r)$. *Then* $\sigma \in \mathbb{S}_q^r$ *iff*

- $\sigma' \in \mathbb{S}_q^r$ *for all proper subsets* $\sigma'$ *of* $\sigma$; *and*
- $\mathbf{b} = \mathbf{a}_i$ *are only solutions* $\mathbf{b} \in \mathcal{N}_q^r = \{\mathbf{c} \in \mathbb{N}^q : |\mathbf{c}| = r\}$ *to:*

$$\mathbf{b} \cdot \mathbf{e}_A \leq \max_i \{\mathbf{a}_i \cdot \mathbf{e}_A\} \quad \textit{for all} \quad A \subseteq [q]$$

*where* $\mathbf{e}_A = \sum_{i \in A} \mathbf{e}_i$



# What does $\mathbb{S}_q^r$ look like: geometric simplifications

Whether or not a set of vertices is a face of $\mathbb{S}_q^r$ is invariant under:

- translation (subtract a common vector)
- permutation of coordinates
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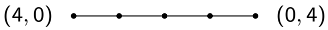
Checking edges reduces to checking $\{0u, v0\}$ where $u, v$ are partitions. This allows us to find, by computer search, all edges for $r \leq 8$ and arbitrary $q$.

# $q \leq 4$

$q = 2$
$(r = 4)$




$q = 3$
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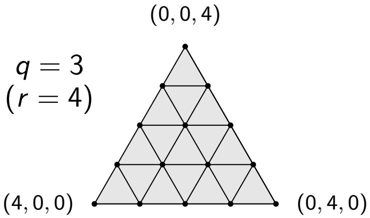

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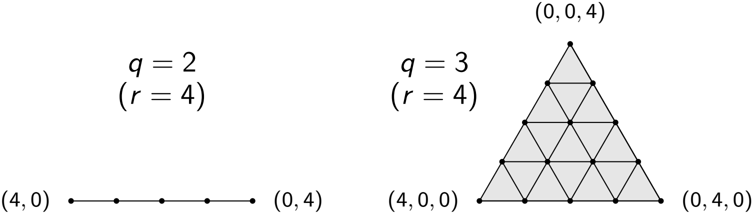


$$U_q^r = \{v \in \{0,1\}^q \colon |v| = r\}$$

- Always a facet
- All facets are translations of $U_q^r$
- Only facet containing $(r, 0, \ldots, 0)$ is $(r-1, 0, \ldots, 0) + U_q^1$
- For $q = 4$, it is drawn in 3-dimensions (octahedron), but it is actually 5-dimensional simplex with 6 vertices. (And similarly for larger $q$.)

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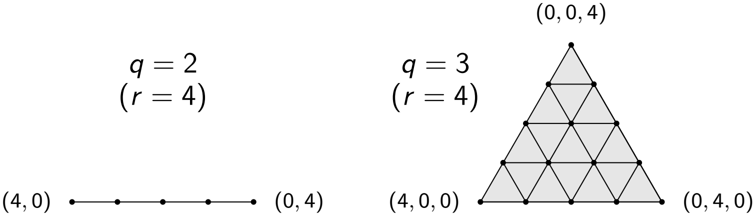


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We can also reflect through a plane, again translating afterwards to restore non-negativity. But in some cases, it may be hard to stay integral.

# Results

**Theorem (EFȘS '24)**

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*$\mathbb{S}_q^r$ supports a minimal free resolution of $\mathcal{E}_q^r$ for $r = 2$ and for $q \leq 4$.*

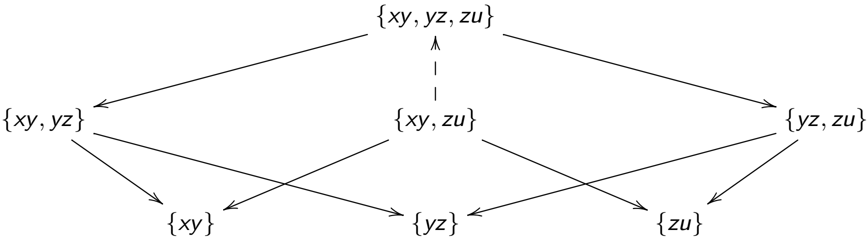
**Theorem (new)**

*$\mathbb{S}_q^3$ supports a minimal free resolution of $\mathcal{E}_q^3$. In particular, it supports a free resolution of $I^3$ for any monomial ideal with $q$ generators.*

**Proof.**

We find a homogeneous acyclic matching [BW '02] (from discrete Morse theory) that leaves exactly the Scarf complex. □

# Homogeneous acyclic matching [BW '02]




- Directed graph of poset of faces of Taylor complex, labeled by lcm
- Partial matching $\mathcal{M}$ using only poset-edges whose poset-vertices have the same label
- Reverse arrows on poset-edges of $\mathcal{M}$
- If new poset is acyclic, then removing $\mathcal{M}$ leaves a complex supporting free resolution

# Enumeration

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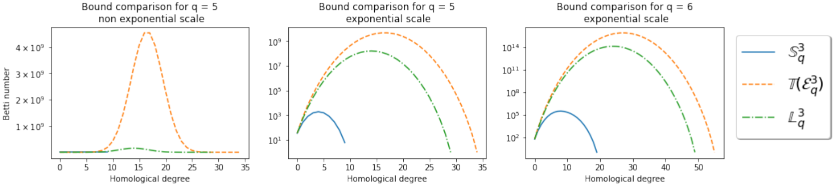
$$\textbf{Taylor: } f_{i-1}(\mathbb{T}(\mathcal{E}_q^3)) = \binom{\binom{q+2}{3}}{i}$$

Both bounds are $O(q^{3i})$, but ...

# Comparing upper bounds



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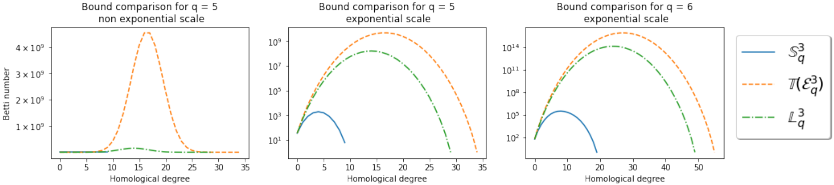


For large $q$

$$\max_i f_i(\mathbb{T}_q^3) / \max_i f_i(\mathbb{S}_q^3) = 2^{q^2 + O(q)} / \sqrt{1 + 6/q}$$

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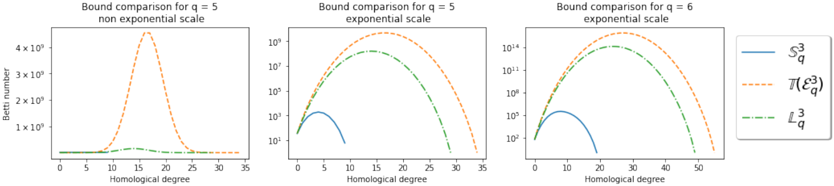


For large $q$; and fixed $0 \leq c < 1$

$$\max_i f_i(\mathbb{T}_q^3) / \max_i f_i(\mathbb{S}_q^3) = 2^{q^2 + O(q)} / \sqrt{1 + 6/q}$$

$$f_{i-1}(\mathbb{T}_q^3)/f_{i-1}(\mathbb{S}_q^3) = (1-c)^{-q^2+O(q)} O(e^{-3cq/(1-c)})$$

where $i = c(\dim(\mathbb{S}_q^3)) + 1$.

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- Further properties of extremal ideal (we are working on this)

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