Recursions for Khovanov-Rozansky Homology and Triangular Catalan Polynomials

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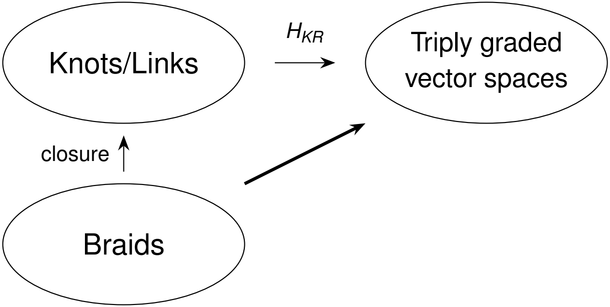
March 28, 2025

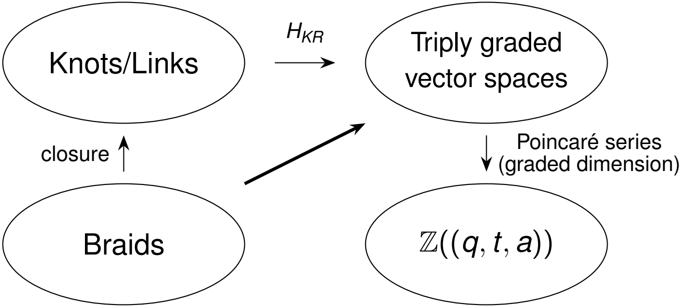
Khovanov-Rozansky homology

$$\text{Knots/Links} \xrightarrow{H_{KR}} \text{Triply graded vector spaces}$$

Khovanov-Rozansky homology



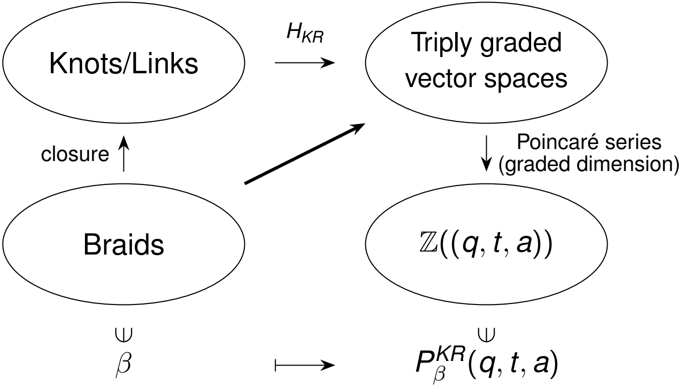
Khovanov-Rozansky homology



Khovanov-Rozansky homology

$$
\begin{tikzcd}
\text{Knots/Links} \arrow[r, "H_{KR}"] & \text{Triply graded vector spaces} \arrow[d, "\text{Poincaré series (graded dimension)}"] \\
\text{Braids} \arrow[u, "\text{closure}"] \arrow[ur] & \mathbb{Z}((q,t,a)) \\
\beta \arrow[u, phantom, "\Cup" description] \arrow[r, mapsto] & P^{KR}_{\beta}(q,t,a) \arrow[u, phantom, "\Cup" description]
\end{tikzcd}
$$

Khovanov-Rozansky homology



Explicit construction, but very hard to compute!

Categorified Young symmetrizers (Elias-Hogancamp)

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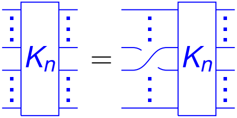
1. 

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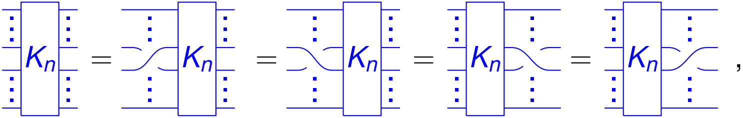
1. 

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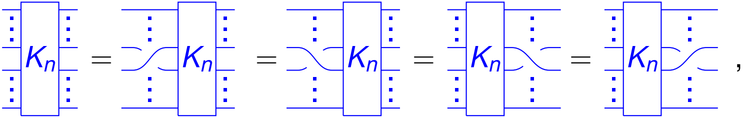
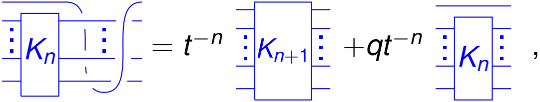
1. 

2. 

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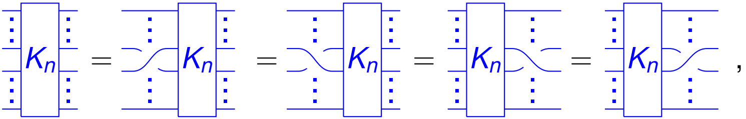
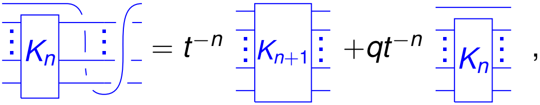
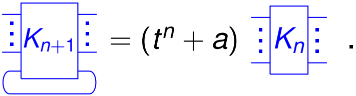
1. 

2. 

3. 

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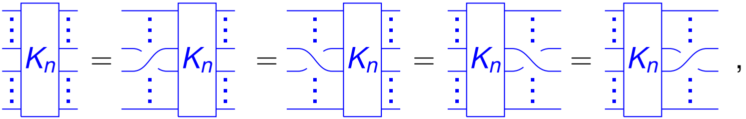
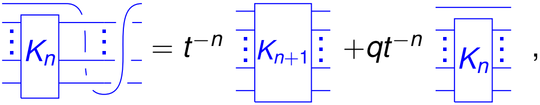
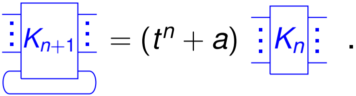
1. 

2. 

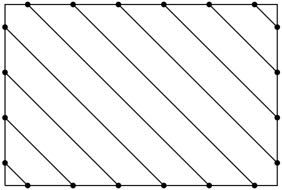
3. 

Remark

In particular, from (2) with $n = 0$ one gets

$$\text{———} = \boxed{K_1} + q\,\text{———} \qquad \textit{or} \qquad (1-q)\,\text{———} = \boxed{K_1}\ .$$

Hogancamp-Mellit recursions



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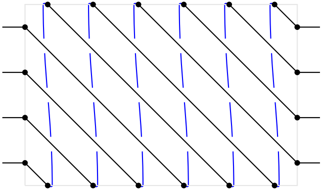

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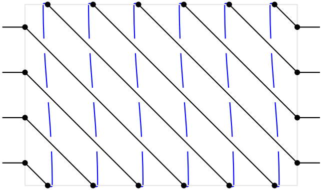

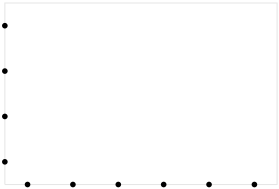

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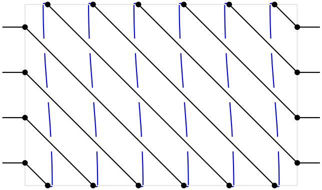

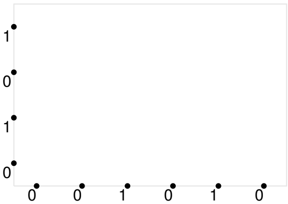

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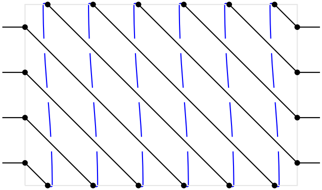

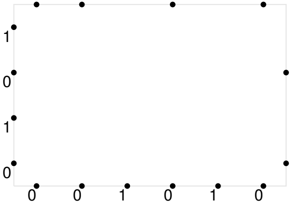

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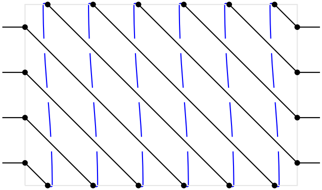

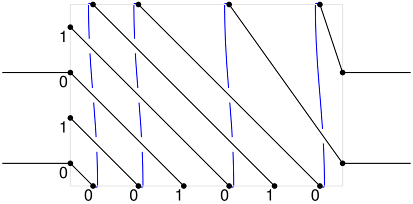

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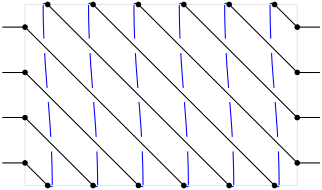

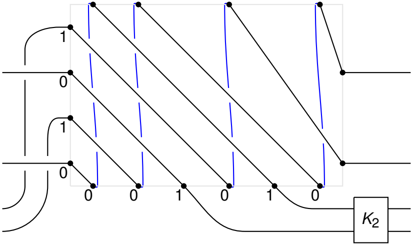

$R_{0101,001010}(q,t,a)$

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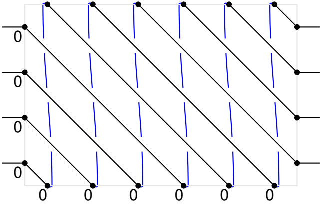

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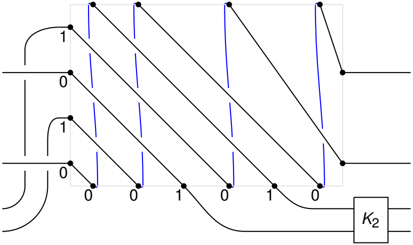

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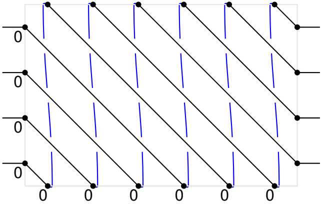

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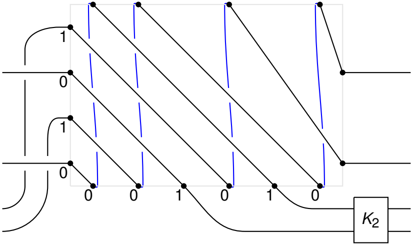

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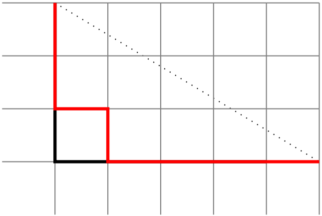
Theorem (Hogancamp-Mellit)

$$R_{0\mathbf{u},0\mathbf{v}} = t^{-|\mathbf{u}|} R_{\mathbf{u}1,\mathbf{v}1} + q t^{-|\mathbf{u}|} R_{\mathbf{u}0,\mathbf{v}0}, \quad R_{1\mathbf{u},0\mathbf{v}} = R_{\mathbf{u}1,\mathbf{v}}, \quad R_{\emptyset,0^n} = \left(\frac{1+a}{1-q}\right)^n,$$

$$R_{1\mathbf{u},1\mathbf{v}} = (t^{|\mathbf{u}|} + a) R_{\mathbf{u},\mathbf{v}}, \qquad R_{0\mathbf{u},1\mathbf{v}} = R_{\mathbf{u},\mathbf{v}1}, \quad R_{0^m,\emptyset} = \left(\frac{1+a}{1-q}\right)^m,$$

where $R_{\emptyset,\emptyset} := 1$.

Rational Schröder polynomials



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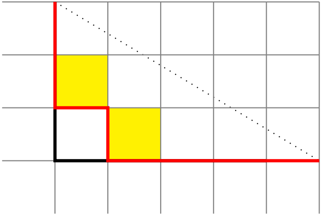

$$S_{n,m}(q,t,a) := \sum_{P \in \mathrm{Dyck}_{n,m}} t^{\mathrm{dinv}(P)} q^{\mathrm{area}(P)} \prod_{\square \in \mathrm{AddBox}(P)} \left(1 + a t^{-\lambda(\square,P)}\right)$$

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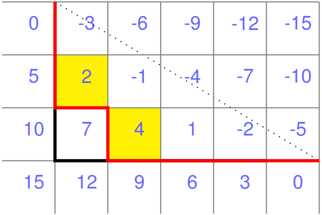

$$\Delta = \mathbb{Z}_{\geq 0} \setminus \{1, 2, 4\} \in \mathrm{Inv}^0_{3,5}$$

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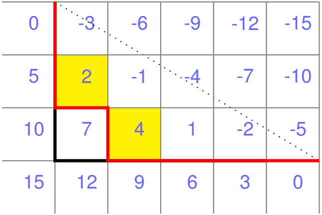

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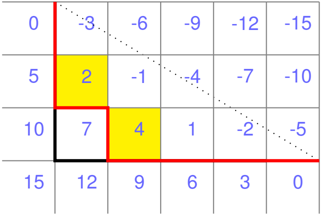

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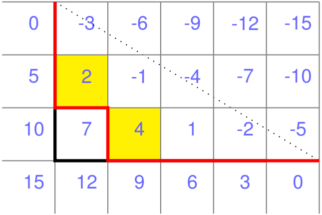

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Gorsky-M-Vazirani recursions

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Let $\mathbf{w} = (w_1, w_2, \ldots, w_{n+m}) \in \{0,1\}^{n+m}$ be a binary sequence.

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$$Q_{\mathbf{w}}(q,t,a) := \sum_{\Delta \in \mathrm{Inv}_{\mathbf{w}}} t^{\mathrm{dinv}(\Delta)} q^{\mathrm{area}(\Delta)} \prod_{k \in \mathrm{cogen}(\Delta)} \left(1 + a t^{-\lambda_k(\Delta)}\right).$$

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Theorem

Let $\boldsymbol{v} = (w_1, \ldots, w_{n+m-1}, 1)$, $\quad \boldsymbol{v'} = (w_1, \ldots, w_{n+m-1}, 0)$. *Then*

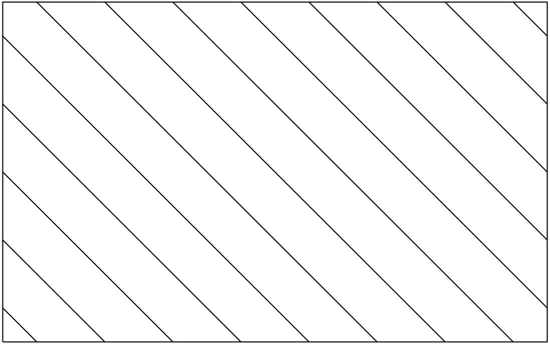
$$Q_{\boldsymbol{w}} = \begin{cases} q(Q_{\boldsymbol{v}} + Q_{\boldsymbol{v'}}), & \text{if } w_0 = w_n = w_m = 0, \\ qQ_{\boldsymbol{v}}, & \text{if } w_0 = 0 \text{ and } w_n + w_m = 1, \\ q\left(1 + a t^{\lambda(\boldsymbol{v})}\right) Q_{\boldsymbol{v}}, & \text{if } w_0 = 0 \text{ and } w_n = w_m = 1, \\ t^{\lambda(\boldsymbol{w})} Q_{\boldsymbol{v}}, & \text{if } w_0 = w_n = w_m = 1. \end{cases}$$

Recursions Match!

Theorem (Gorsky-Mazin-Vazirani)

$$S_{n,m}(q,t,a) = \frac{1-q}{q^{n+m}} Q_{0\ldots0}(q,t,a) = (1-q)t^{\frac{(n-1)(m-1)}{2}} R_{0\ldots0,0\ldots0}(q,t^{-1},a).$$

Shortcut torus knots



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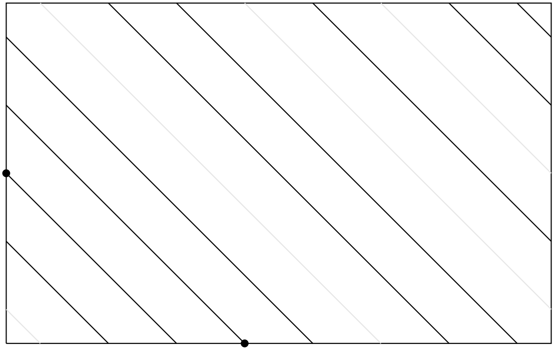

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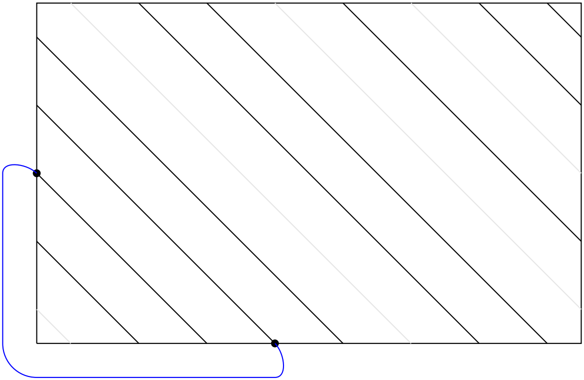

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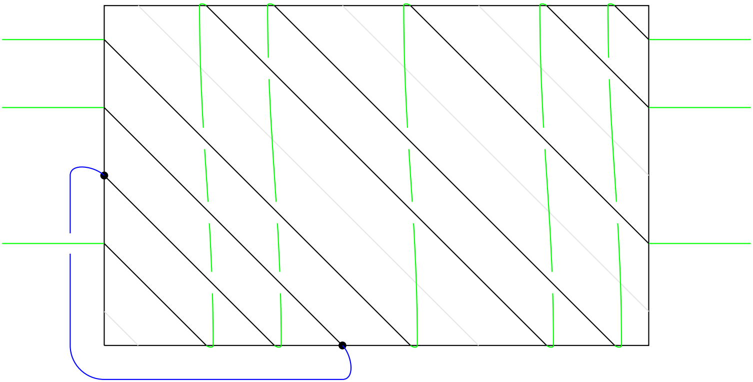

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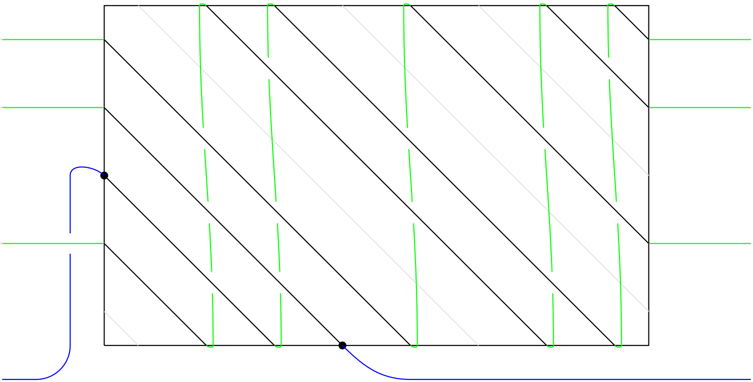

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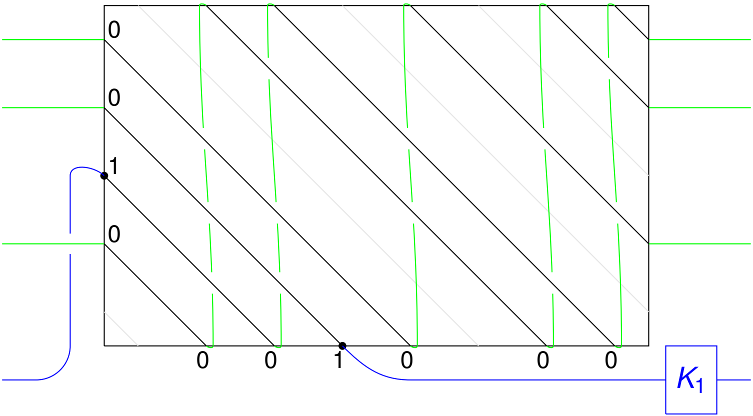

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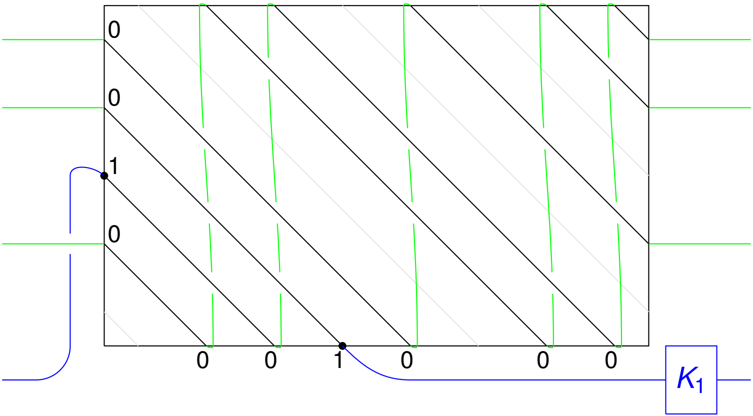
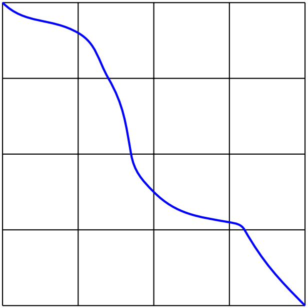


Theorem

All knots that appear in the Hogancamp-Mellit recursion are the shortcut torus knots (as above).

Monotone knots of Galashin-Lam



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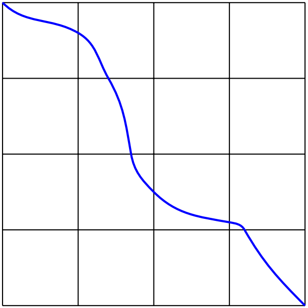

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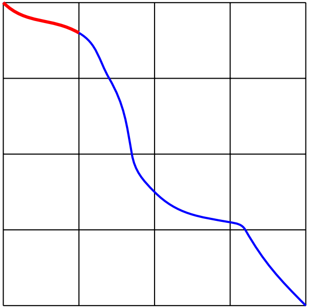

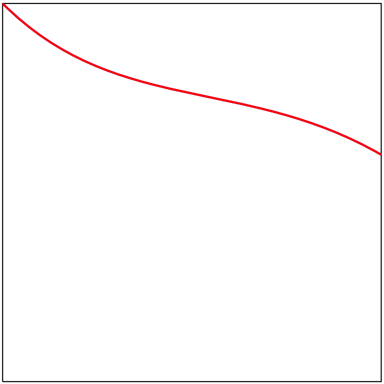

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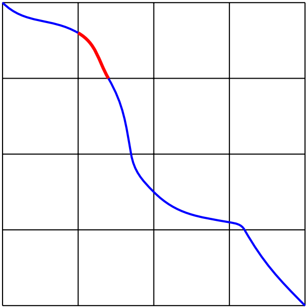

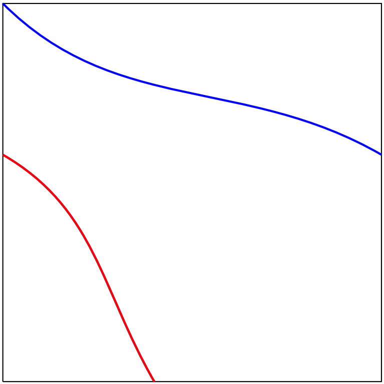

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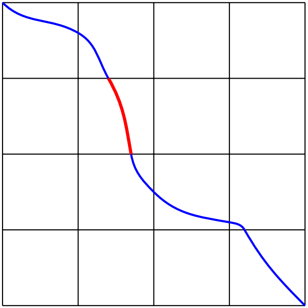

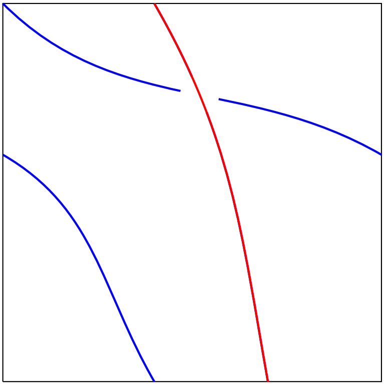

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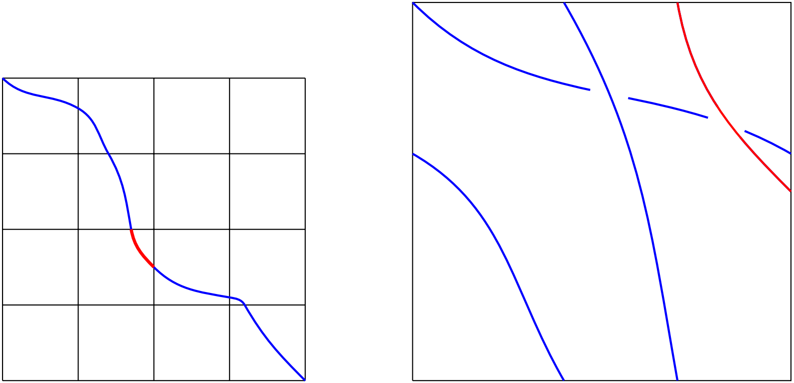
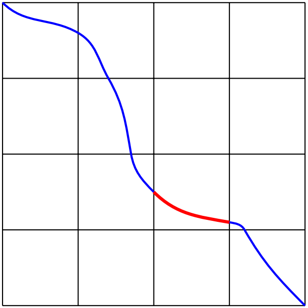
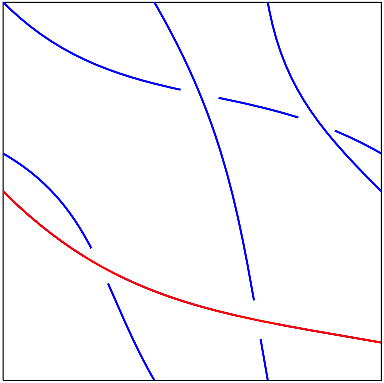
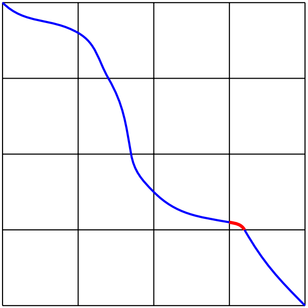


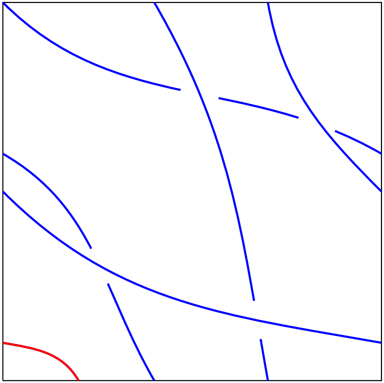
Monotone knots of Galashin-Lam



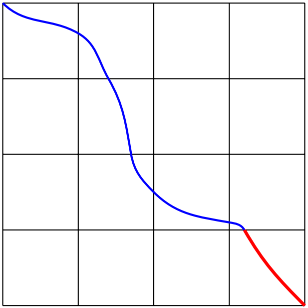


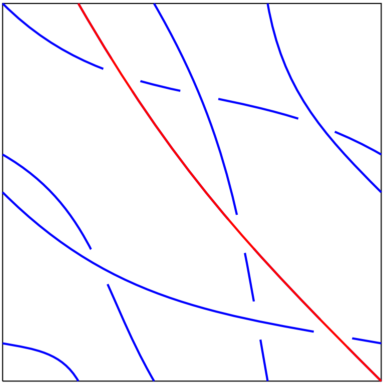
Monotone knots of Galashin-Lam



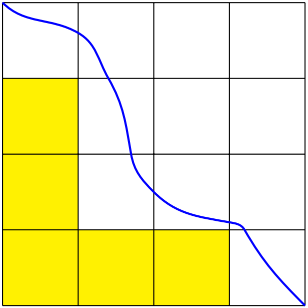


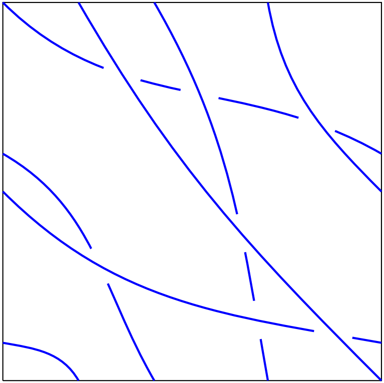
Monotone knots of Galashin-Lam





Monotone knots of Galashin-Lam

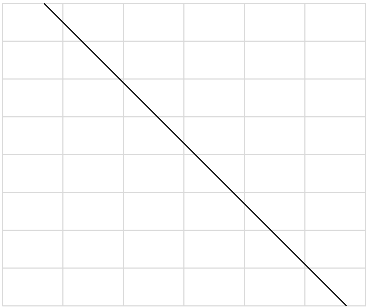


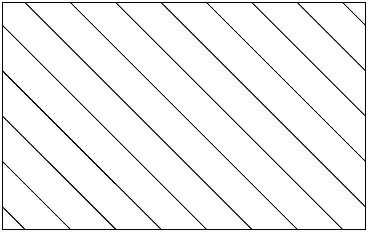


Theorem (Galashin-Lam)

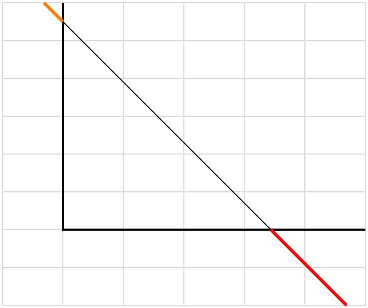
Up to an isotopy, the knot only depends on the partition under the curve.

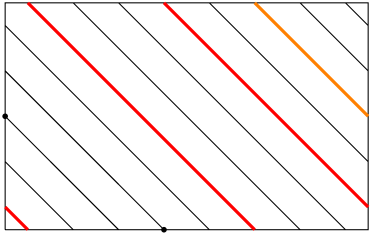
Monotone Knots of Triangular Partitions



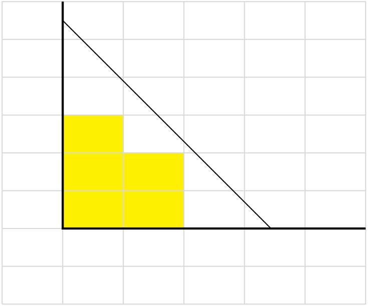


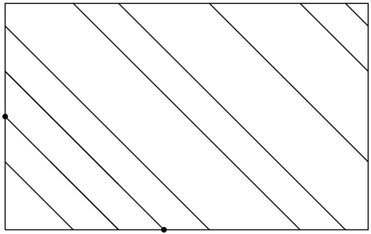
Monotone Knots of Triangular Partitions



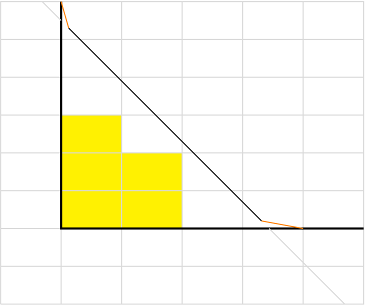


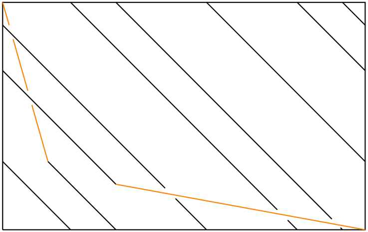
Monotone Knots of Triangular Partitions



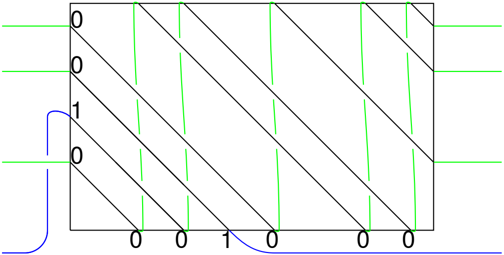


Monotone Knots of Triangular Partitions





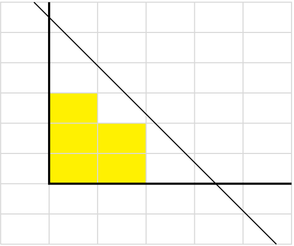
Monotone Knots of Triangular Partitions



Theorem

Monotone knots of the triangular partitions are the shortcut torus knots.

Triangular Schröder polynomials



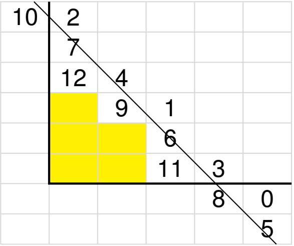
τ - Triangular partition

($\tau = (3, 2)$ in the example)

Theorem

$$S_\tau(q,t,a) := \sum_{P \in \mathrm{Dyck}(\tau)} t^{\mathrm{dinv}(P)} q^{\mathrm{area}(P)} \prod_{\square \in \mathrm{AddBox}(P)} \left(1 + a t^{-\lambda(\square, P)}\right)$$

Triangular Schröder polynomials



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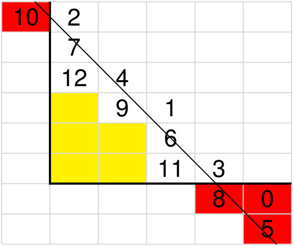
Triangular Schröder polynomials



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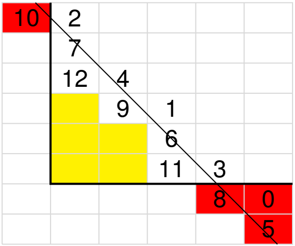


τ - Triangular partition

($\tau = (3, 2)$ in the example)

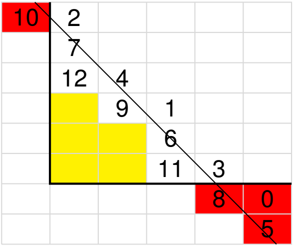
$0, 5, 8, 10$ are forced in, the rest are out

$\mathbf{w} = 1000010010100$

Theorem

$$S_\tau(q, t, a) := \sum_{P \in \mathrm{Dyck}(\tau)} t^{\mathrm{dinv}(P)} q^{\mathrm{area}(P)} \prod_{\square \in \mathrm{AddBox}(P)} \left(1 + a t^{-\lambda(\square, P)}\right)$$

Triangular Schröder polynomials



τ - Triangular partition

($\tau = (3, 2)$ in the example)

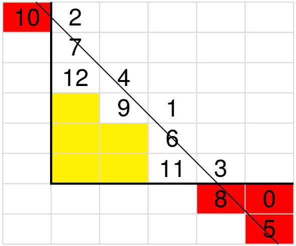
$0, 5, 8, 10$ are forced in, the rest are out

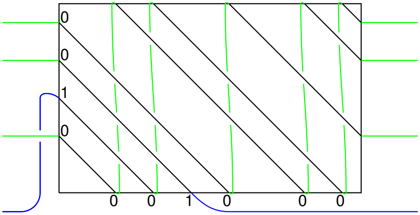
$\mathbf{w} = 1000010010100$

Theorem

$$S_\tau(q,t,a) := \sum_{P \in \mathrm{Dyck}(\tau)} t^{\mathrm{dinv}(P)} q^{\mathrm{area}(P)} \prod_{\square \in \mathrm{AddBox}(P)} \left(1 + at^{-\lambda(\square, P)}\right)$$
$$= Q_{\mathbf{w}}(q,t,a).$$

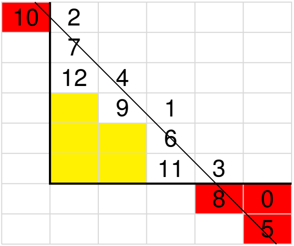
Back to Hogancamp-Mellit Recursion

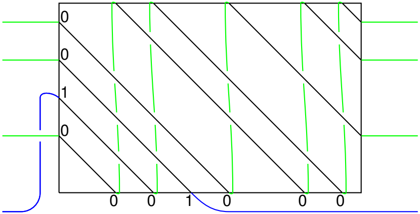




$\mathbf{u} = 0100$, $\mathbf{v} = 001000$

Back to Hogancamp-Mellit Recursion





$\mathbf{u} = 0100$, $\mathbf{v} = 001000$

Theorem

$$S_\tau(q, t, a) = Q_{\mathbf{w}}(q, t, a) = t^{|\tau|} R_{\mathbf{u},\mathbf{v}}(q, t^{-1}, a) = t^{|\tau|} P^{KR}_{\beta(\tau)}(q, t^{-1}, a).$$

Thank you!