Dual Mixed Volume of Polytopes

Joint work with Yibo Gao and Thomas Lam

Lei Xue

University of Michigan

leixue@umich.edu

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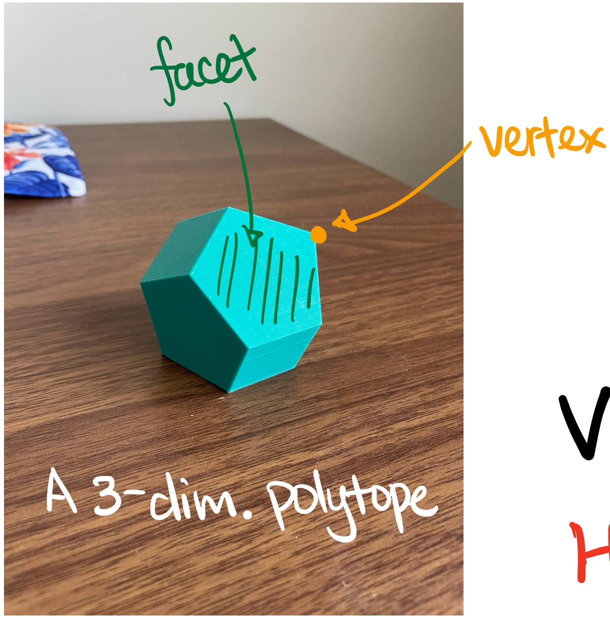
Polytopes : two (equivalent) definitions



V-polytope: convex hull of finitely many pts.

H-polytope: bounded intersection of half-spaces

Let P be d-dim. polytope in $\mathbb{R}^d$.

- Support function

$$\begin{aligned} h_P : \mathbb{R}^d &\longrightarrow \mathbb{R} \\ \vec{v} &\longmapsto -\min_{\vec{p} \in P} \langle \vec{v}, \vec{p} \rangle \end{aligned}$$

Let P be d-dim. polytope in $\mathbb{R}^d$.

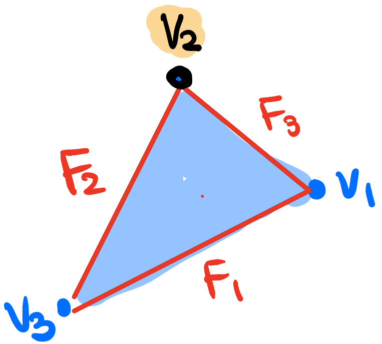
- **Support function**

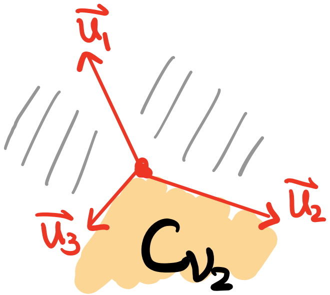
$$h_P : \mathbb{R}^d \longrightarrow \mathbb{R}$$
$$\vec{v} \longmapsto -\min_{\vec{p}\in P} \langle \vec{v}, \vec{p} \rangle$$

Remarks
- h_P is piecewise linear
- $P = \{\vec{y} \mid \langle \vec{v}, \vec{y} \rangle \leq h_P(\vec{v}) \ \forall \vec{v}\}$

- **Normal fan $N(P)$** consists of the cones:

$$C_F := \{\vec{v} \in \mathbb{R}^d \mid h_P(\vec{v}) = -\langle \vec{v}, \vec{y} \rangle \quad \forall \vec{y} \in F\}$$





(polar) Dual Polytopes

Given a d-polytope in $\mathbb{R}^d$, its polar dual is

$$P^\vee = \{\vec{x} \in \mathbb{R}^d \mid h_P(\vec{x}) \leq 1\} = \{\vec{x} \in \mathbb{R}^d \mid \langle \vec{x}, \vec{p} \rangle \geq -1 \text{ for ALL } \vec{p} \in P\}$$

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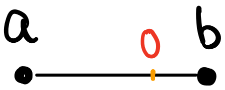
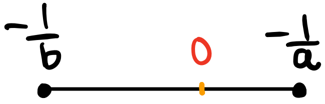
Ex. 0. $P =$ 

, $P^\vee =$ 

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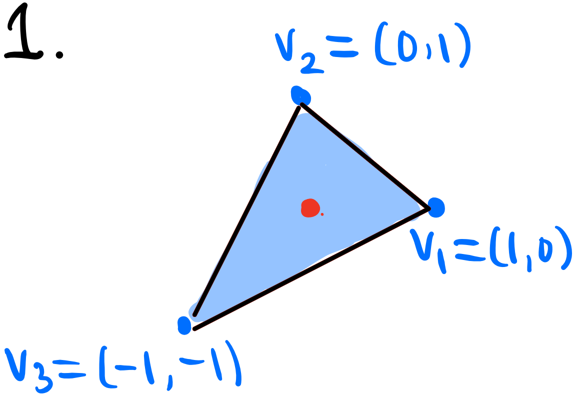
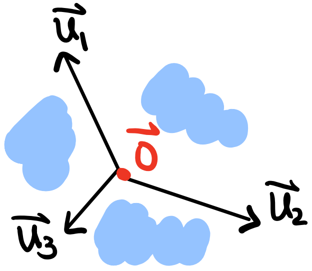
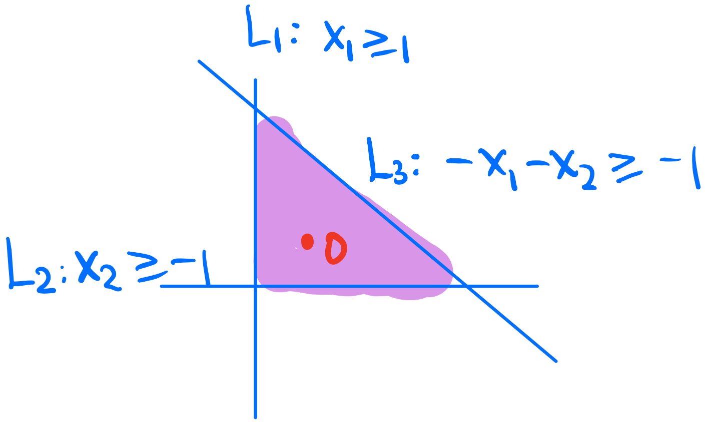
Ex. 1.







P $\qquad$ $N(P)$ $\qquad$ $P^\vee$

vertices of P $\sim$ maximal cones of $N(P)$ $\sim$ facets of $P^\vee$

(polar) Dual Polytopes

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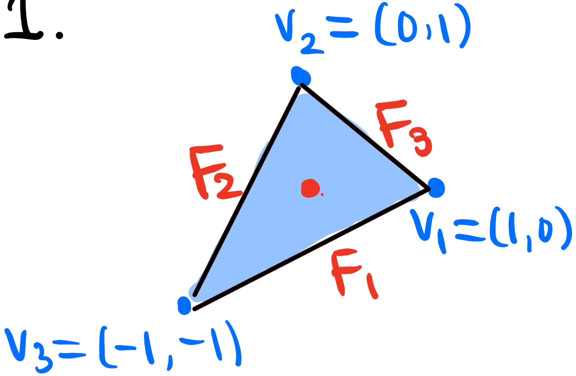
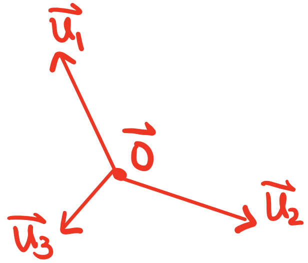
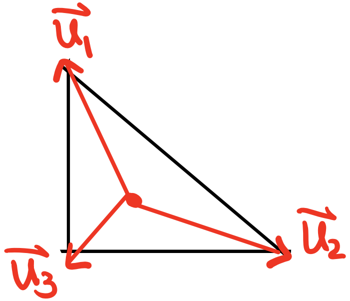




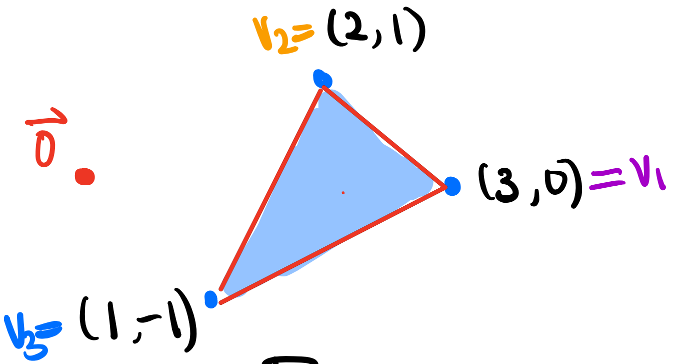


P $\qquad$ $N(P)$ $\qquad$ $P^\vee$

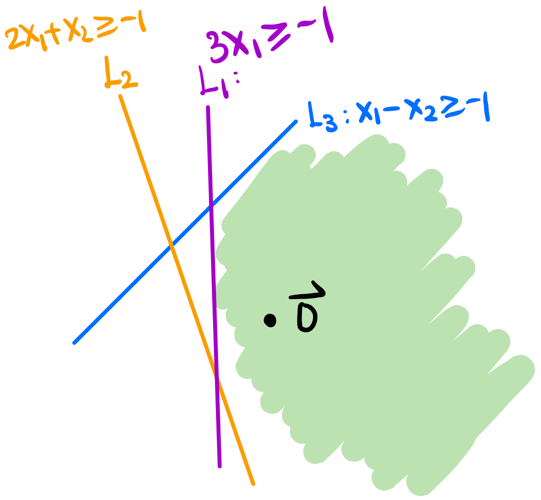
facets of P ~ rays in $N(P)$ ~ vertices of $P^\vee$

(polar) Dual Polytopes

Remark: P is a polytope with $\vec{0}$ in its interior if and only if $P^\vee$ is also a polytope, in which case $P^{\vee\vee} = P$.



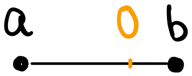
P



$P^\vee$: an (unbounded) polyhedron.

Dual Volume of Polytopes

Ex. 0

$P =$ 

, $P^\vee =$ 

, $\mathrm{Vol}(P^\vee) = \left(-\frac{1}{a}\right) - \left(-\frac{1}{b}\right) = \frac{a-b}{ab}.$

Dual Volume of Polytopes

Ex. 0

$P =$ 

,

Dual Volume of Polytopes

Ex. 0

$$P = \overset{a}{\bullet}\!\!-\!\!-\!\!\overset{x}{\cdot}\!\!-\!\!-\!\!\overset{b}{\bullet}\ ,$$

$$P - x = \overset{a-x}{\bullet}\!\!-\!\!-\!\!\overset{0}{\cdot}\!\!-\!\!-\!\!\overset{b-x}{\bullet}\ ; \qquad (P-x)^\vee = \overset{\frac{1}{x-b}}{\bullet}\!\!-\!\!-\!\!\overset{0}{\cdot}\!\!-\!\!-\!\!\overset{\frac{1}{x-a}}{\bullet}$$

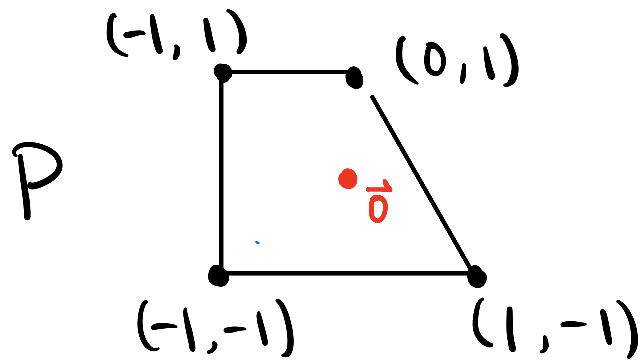
$$\mathrm{Vol}\left((P-x)^\vee\right) = \frac{1}{x-a} - \frac{1}{x-b} = \frac{b-a}{(x-a)(x-b)}$$

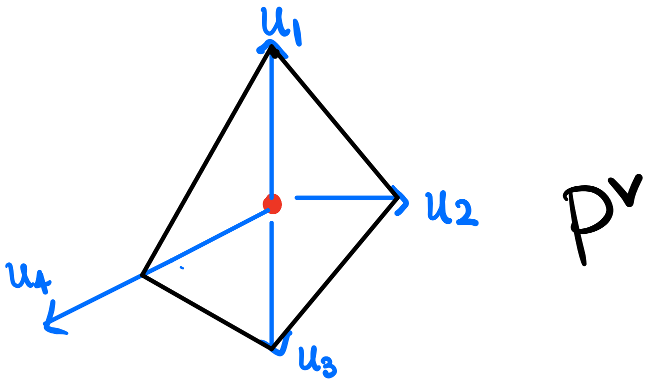
The dual volume function of P

(denoted by $f_P(x)$)

Dual Volume of Polytopes

Ex. 2



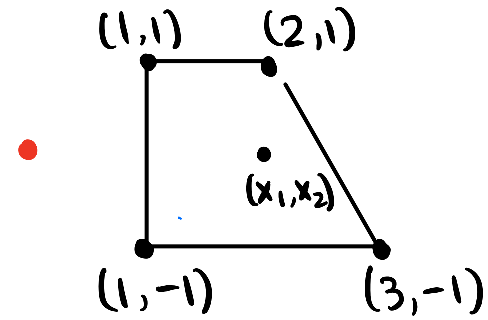


$$\mathrm{Vol}(P^\vee) = 1 + 1 + \frac{2}{5} + \frac{2}{5} = \frac{14}{5}.$$

Dual Volume of Polytopes

Ex. 2^*

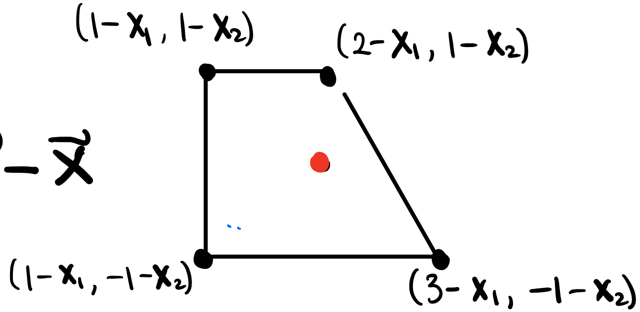
P



Dual Volume of Polytopes

Ex. 2^*

$P - \vec{x}$



Dual Volume of Polytopes

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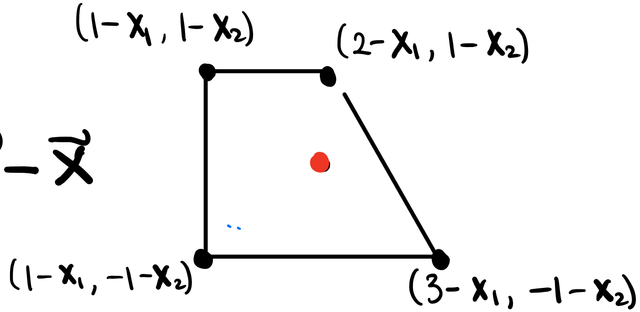

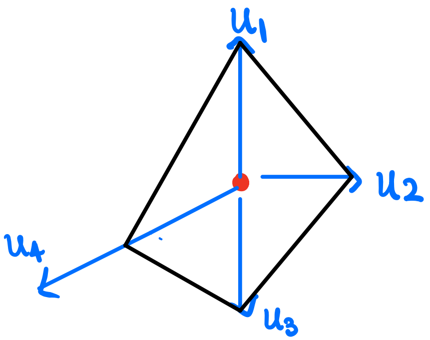

$(P-\vec{x})^\vee$

$$f_P(\vec{x}) = \mathrm{Vol}\left((P-\vec{x})^\vee\right) = \frac{1}{(1+x_2)(-1+x_1)} + \frac{1}{(-1+x_1)(1-x_2)} + \frac{2}{(1-x_2)(5-2x_1-x_2)} + \frac{2}{(5-2x_1-x_2)(1+x_2)}$$

$$= \frac{6-x_2}{(1+x_2)(-1+x_1)(1-x_2)(5-2x_1-x_2)}$$

Remarks:

1. $f_P(\vec{x})$ is a rational function in $x_1, \cdots, x_d$.
2. Written as one fraction:

   Denominator: product of linear factors, each corresponding to a facet of P.

   Numerator: coincides with the adjoint polynomial of P

Dual Volume Function

Def. (Gao, Lam, X.)

Let P be a non-degenerate polyhedron, $\mathcal{T} = \{C_1, \cdots, C_N\}$ a triangulation of its normal fan $N(P)$, the dual volume function of P is

$$f_P(\vec{x}) := \sum_{\substack{C = \operatorname{cone}\{\vec{u}_1, \cdots, \vec{u}_d\}, \\ C \in \mathcal{T}}} \frac{|\det(\vec{u}_1, \cdots, \vec{u}_d)|}{\prod_{i=1}^{d} h_{P-\vec{x}}(\vec{u}_i)}$$

Remarks: 1. $f_P(\vec{x})$ does NOT depend on $\mathcal{T}$.

2. If $\vec{x} \in \operatorname{int}(P)$, then $f_P(\vec{x}) = \operatorname{Vol}((P-\vec{x})^\vee)$.

Dual Volume Function

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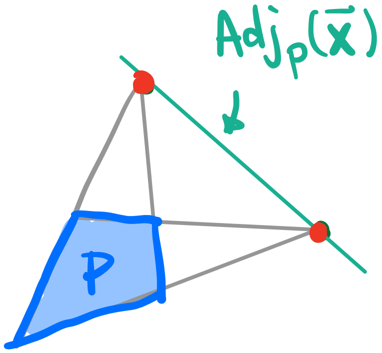
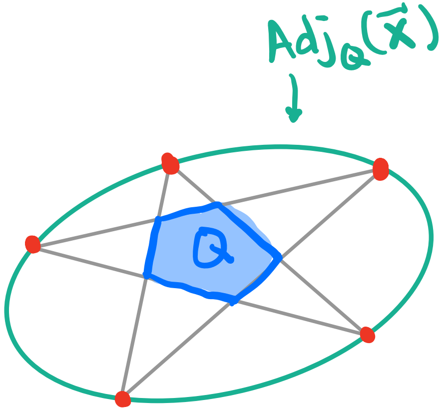
$$f_P(\vec{x}) := \sum_{\substack{C = \mathrm{cone}\{\vec{u}_1, \cdots, \vec{u}_d\}, \\ C \in \mathcal{J}}} \frac{|\det(\vec{u}_1, \cdots, \vec{u}_d)|}{\prod_{i=1}^{d} h_{P-\vec{x}}(\vec{u}_i)} = \frac{\mathrm{Adj}_P(\vec{x})}{\prod_{F_i : \text{facets of } P} h_{P-\vec{x}}(\vec{u}_i)}$$

$\mathrm{Adj}_P(\vec{x})$: the adjoint of P.

[Warran, 1996], [Kohn-Ranestad, 2020]





Dual Mixed Volume

Let P_1, P_2 be two polytopes in $\mathbb{R}^d$. Their **Minkowski sum** is the following polytope.

$$P_1 + P_2 = \{\vec{p}_1 + \vec{p}_2 \mid \vec{p}_1 \in P_1, \ \vec{p}_2 \in P_2\}$$

Dual Mixed Volume

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$$P_1 + P_2 = \{\vec{p}_1 + \vec{p}_2 \mid \vec{p}_1 \in P_1,\ \vec{p}_2 \in P_2\}$$

For a sequence of polytopes $P_\bullet = (P_1, P_2, \cdots, P_r)$ in $\mathbb{R}^d$, $\vec{x} = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r$, if each P_i is non-degenerate and $\vec{0} \in \mathrm{int}(x_1P_1 + \cdots + x_rP_r)$, the dual mixed volume function is

$$m_{P_\bullet}(x_1, \cdots, x_r) = \mathrm{Vol}\left((x_1P_1 + \cdots + x_rP_r)^\vee\right)$$

Dual Mixed Volume

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$$m_{P_\bullet}(x_1, \cdots, x_r) = \mathrm{Vol}\left((x_1P_1 + \cdots + x_rP_r)^\vee\right)$$

Remarks: $m_{P_\bullet}$ is a rational function in $x_1, \cdots, x_r$ with degree $-d$.

$m_{P_\bullet}$ generalizes $f_P(\vec{x})$

denominator is a product of linear factors, each corresponding to a facet of $x.P.$

Dual Mixed Volume function

Def: Let $P_\bullet = (P_1, \cdots, P_r)$ be a regular* sequence of polyhedra w/. normal fan $N(P)$,
Define $h_{xP_\bullet} = x_1 h_{P_1} + \cdots + x_r h_{P_r}$.
For any triangulation $\mathcal{J}$ of $N(P)$, C is a simplicial cone in $\mathcal{J}$ generated by the vectors $\vec{u}_1, \cdots, \vec{u}_d$, and let $\det(C) = \det(\vec{u}_1, \cdots, \vec{u}_d)$. The dual mixed volume function is

$$m_{P_\bullet}(\vec{x}) = \frac{\sum_{C \in \mathcal{J}} |\det(C)| \prod_{\substack{\vec{u}:\text{ rays of } N(P) \\ \vec{u} \notin C}} h_{xP_\bullet}(\vec{u})}{\prod_{\vec{u}:\text{ rays of } N(P)} h_{xP_\bullet}(\vec{u})}$$

* $h_{xP_\bullet} \neq 0$ in $N(P) \setminus \vec{0}$.

Motivation ... from mixed volume

For a sequence of convex bodies $S_\bullet = (S_1, \cdots, S_r)$ in $\mathbb{R}^d$ and $x_1, \cdots, x_r > 0$,

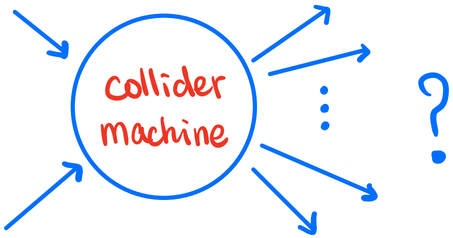
The **mixed volume polynomial**:

$$\mathrm{Vol}(x_1 S_1 + \cdots + x_r S_r) = \sum \underbrace{V(S_{i_1}, \cdots, S_{i_d})}_{\text{mixed volume of } S_{i_1}, \cdots, S_{i_d}} x_{i_1} x_{i_2} \cdots x_{i_d}$$

Alexandrov-Fenchel inequality [Minkowski 1903][Alexandrov 1938]

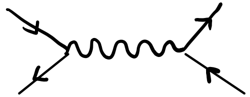
$$V(S_1, S_2, S_3, \cdots, S_d)^2 \geq V(S_1, S_1, S_3, \cdots, S_d) \cdot V(S_2, S_2, S_3, \cdots, S_d).$$

Motivation ... from quantum physics



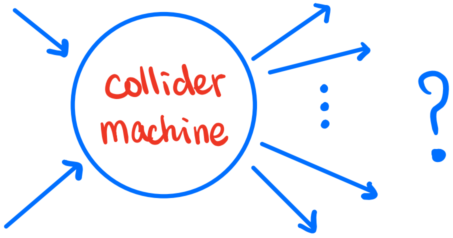
Scattering of elementary particles:

What is the probability of possible outcomes?



Feymann diagram

Motivation ... from quantum physics



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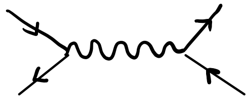


Feynmann diagram

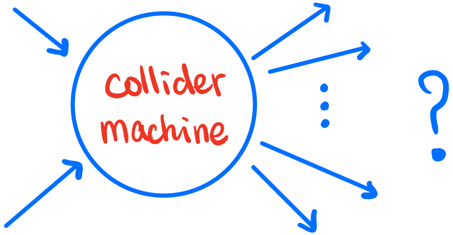
Scattering amplitude $\Omega\, d\vec{x}$

Rational function* used to predict outcomes of experiments.

Constrained by "information about where the poles and residues are"

* At tree level

Motivation ... from quantum physics



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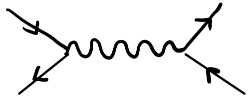

Feynman diagram

Scattering amplitude $\Omega \, d\vec{x}$

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↓ match with

Combinatorics of faces and boundaries of polytopes

* At tree level

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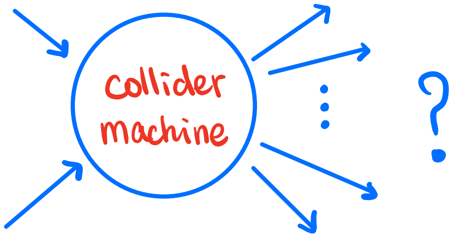

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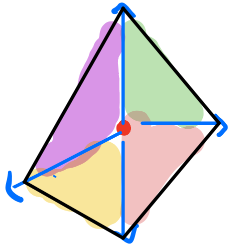


Scattering amplitude $\Omega\, d\vec{x} = f_P(\vec{x})\ d\vec{x}$

Rational function* used to predict outcomes of experiments.

Constrained by "information about where the poles and residues are"

↓ match with

"Positive Geometry"

Combinatorics of faces and boundaries of polytopes

[Arkani-Hamed, Trnka, 2013], [ABL, 2017]

* At tree level

Our Findings:

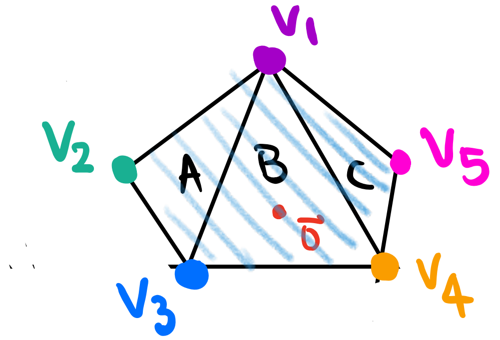
Properties and formulae

- Integral formula

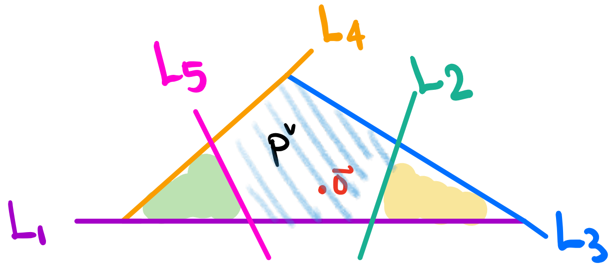
Our Findings:

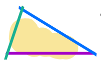
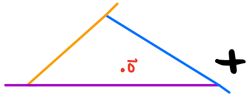
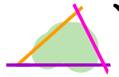
Properties and formulae

- Integral formula
- Dual volume is valuative! (generalizing [Filliman '92] and [Kuperburg '03].)



$$[P] = [A] + [B] + [C]$$



$\mathrm{Vol}(P^\vee) = (-$  $) +$  $+ (-$  $)$

Our Findings:

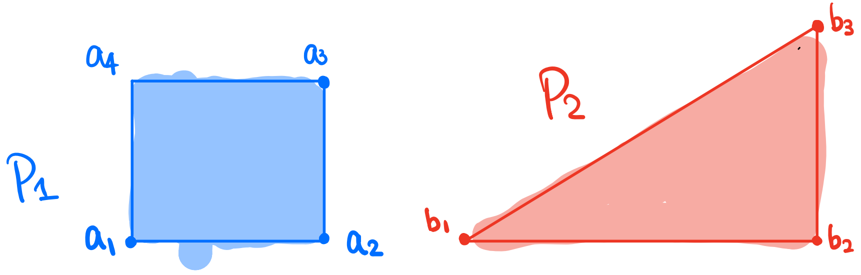
Properties and formulae

- Integral formula
- Dual volume is valuative!
- Dual mixed volume is preserved under mixed subdivisions.

(via the Cayley trick)

The Cayley Trick [Sturmfels '94] [Huber, Rambau, Santos, '00]

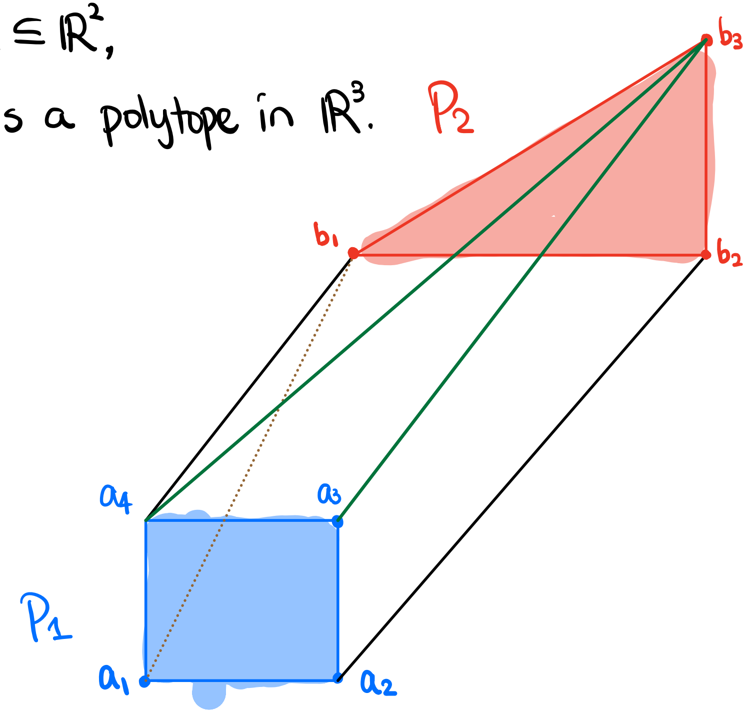
For $P_1, P_2 \subseteq \mathbb{R}^2$,



The Cayley Trick

For $P_1, P_2 \subseteq \mathbb{R}^2$,

$C(P_1, P_2)$ is a polytope in $\mathbb{R}^3$.



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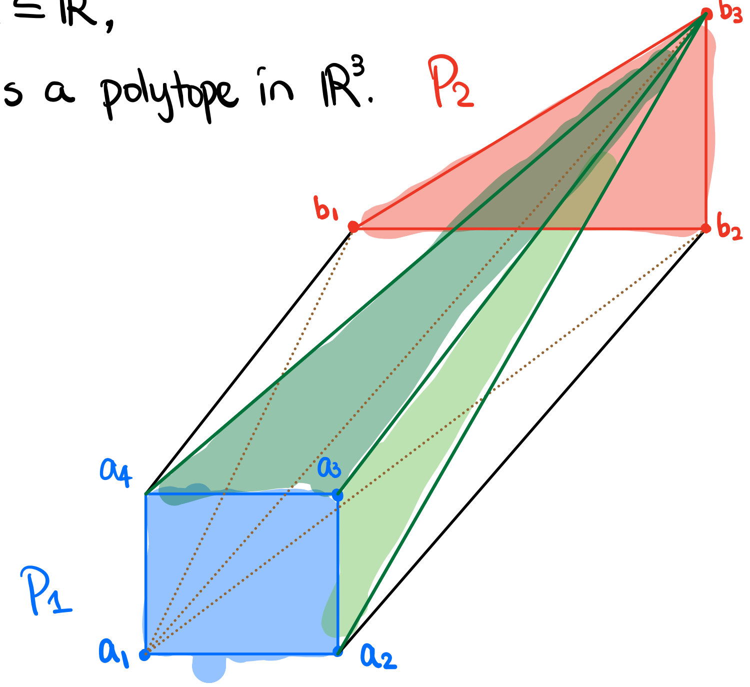

A subdivision of $C(P_1, P_2)$:

$\{a_1a_2a_3a_4b_3,\ a_1b_1b_2b_3,\ a_1a_2b_2b_3,\ a_1a_4b_1b_3\}$

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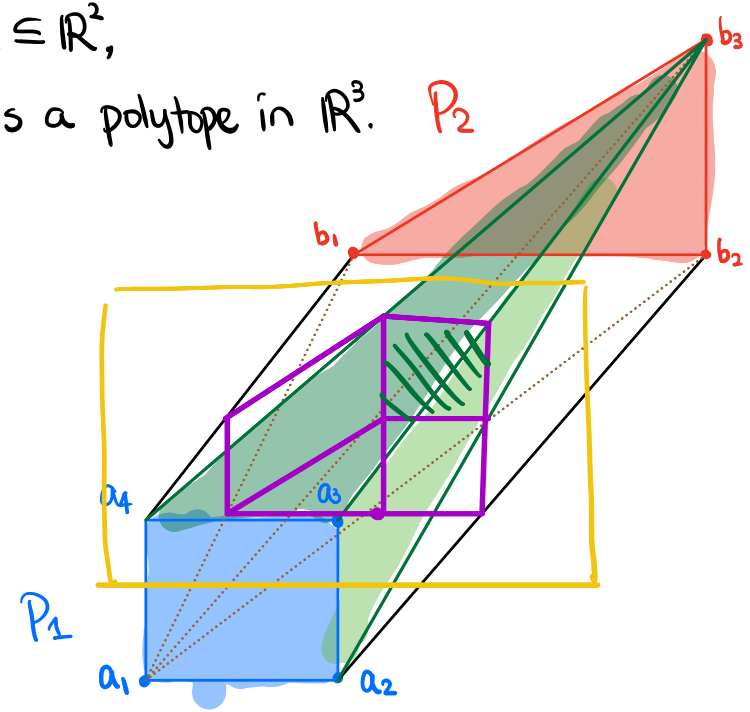
For $P_1, P_2 \subseteq \mathbb{R}^2$,

$C(P_1, P_2)$ is a polytope in $\mathbb{R}^3$.

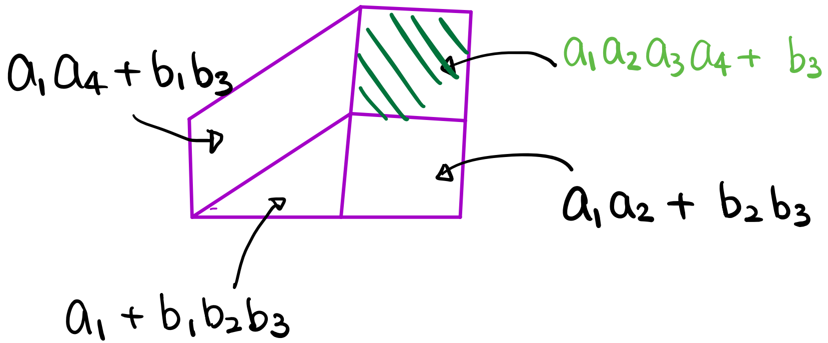


A subdivision of $C(P_1, P_2)$:

$\{a_1a_2a_3a_4b_3,\ a_1b_1b_2b_3,\ a_1a_2b_2b_3,\ a_1a_4b_1b_3\}$

$\Updownarrow$

Mixed subdivision of $x_1P_1 + x_2P_2$



Other Formulae:

- Zonotopes (Deletion-Contraction)
- Generalized permutahedra
- Associahedra ⇝ Scattering Amplitude*

Questions / Future directions:

- the numerator of $m_{P}(\vec{x})$:

  when are the coefficients positive?

- discrete version? ("dual mixed Ehrhart polynomial"?) [Lam, 2025+]

*φ^3-planar amplitude at tree level

Thank you!