IDP for 2-Partition Maximal Polytopes

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Matt Mills
307
Higher
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Algebraic
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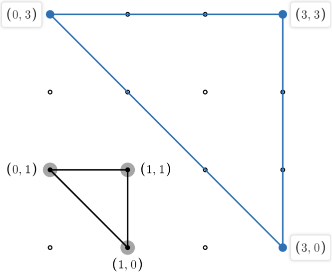
Example



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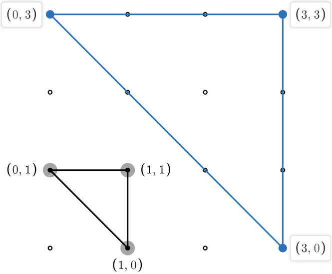


Definition

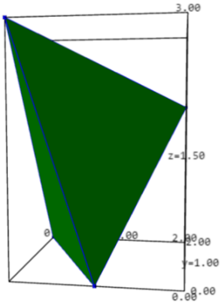
A (lattice) polytope P has the **integer decomposition property** (IDP) provided for all $t \in \mathbb{Z}^{>0}$ and (integer point) $x \in tP$, there exists $x_1, x_2, \ldots, x_t \in P$ so that

$$x = x_1 + x_2 + \cdots + x_t.$$

That is, the integer points in the t^{th} dialation of P is a sum of t points in P.

Example

$P = \mathrm{conv}(\ (1,0,0),(0,2,0),(0,0,3),(2,2,2)\ )$



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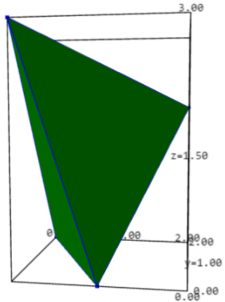

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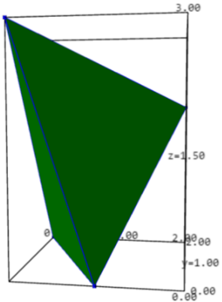

$(1,1,2), (1,2,1) \in P$

$(1,1,3) \in 2P$

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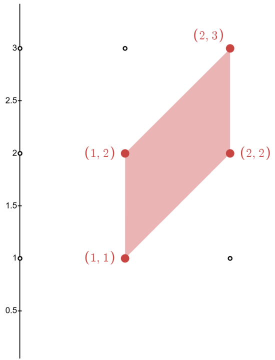

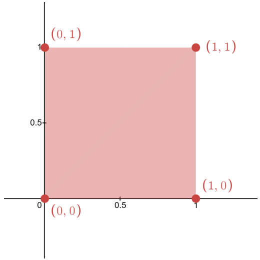

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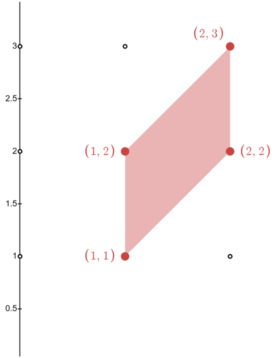

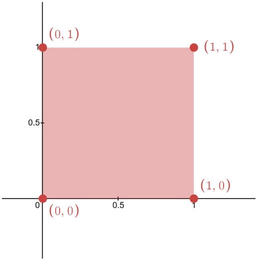

How can we build symmetric polytopes in general?

Given a partition λ, one can define its **Schur function** $s_\lambda \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ which has the form

$$s_\lambda(x_1, \ldots, x_n) = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ corresponds to a *semistandard tableaux of shape* λ *and content* α.

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If $\lambda = (3, 2)$ and $\alpha = (1, 2, 2)$ we could have

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(Rows *weakly* increase, columns *strictly* increase.)

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So

$$s_\lambda(x_1, x_2, x_3) = 2x_1 x_2^2 x_3^2 + \cdots$$

Example

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\begin{aligned}
s_{(3,2)} &= x_1^3x_2^2 + x_1^3x_3^2 + x_1^2x_2^3 + x_1^2x_3^3 + x_2^2x_3^3 + x_2^3x_3^2 \\
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Sounds like our desired property for symmetric polytopes...

Definition

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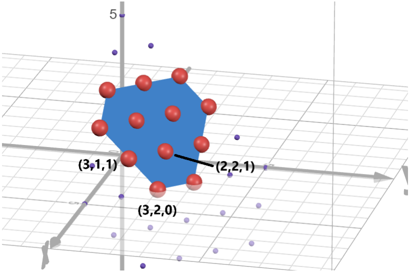

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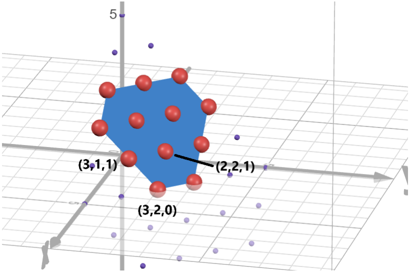

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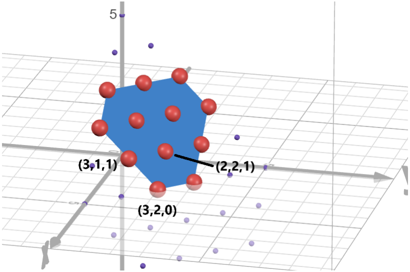

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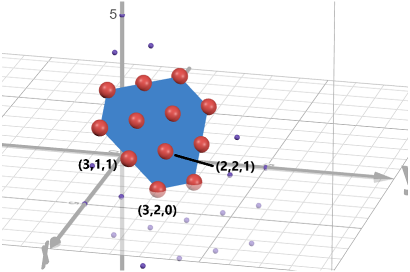

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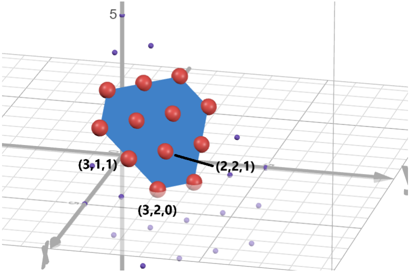

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- **2-Partition Maximal Polytopes** (2025)

Definition

If μ and λ are partitions, we say $\mu \leq \lambda$ when

$$\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i$$

for every k. If neither $\mu \leq \lambda$ nor $\lambda \leq \mu$, they are **pairwise maximal**.

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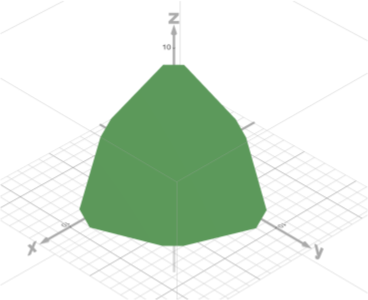

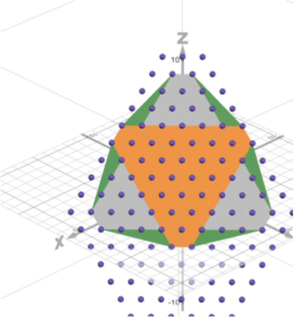

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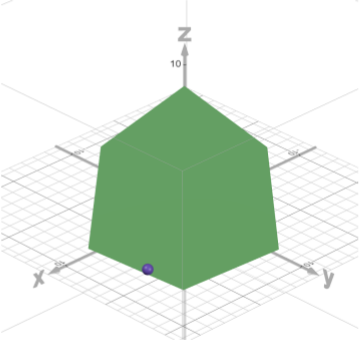

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Let $M = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the set of pairwise maximal points in a polytope. Let

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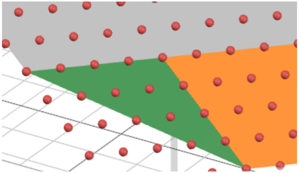

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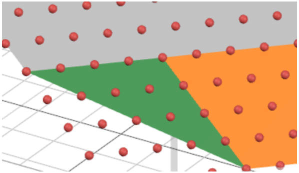

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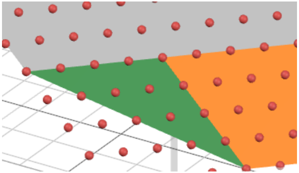

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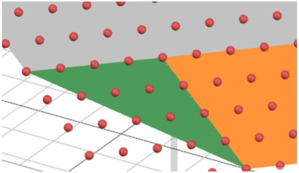

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1. Different lengths for partitions.
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3. Multiple maximal partitions.