Characterizing Weyl Alternation Sets for Roots of the Type A Lie Algebra

Kimberly P. Hadaway*
Iowa State University

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Portia X. Anderson, Esther Banaian, Melanie J. Ferreri,
Owen C. Goff, Pamela E. Harris, Kimberly J. Harry,
Nicholas Mayers, Shiyun Wang, and Alexander N. Wilson

Outline

- Background with Definitions and Examples
- Motivation
- Main Results

Integer Partitions

Recall: An integer partition of a **positive** integer n is a tuple of numbers (ordered in weakly decreasing order) whose sum equals n.

Example

Let $n = 5$:

5	(5)	$4+1$	$(4,1)$
$3+2$	$(3,2)$	$3+1+1$	$(3,1,1)$
$2+2+1$	$(2,2,1)$	$2+1+1+1$	$(2,1,1,1)$
$1+1+1+1+1$	$(1,1,1,1,1)$		

Vector Partition Functions

Let A be an $m \times d$ integral matrix.

Goal: compute the value of the vector partition function

$$\phi_A(\mathbf{b}) = \#\{\mathbf{x} \in \mathbb{N}^d \ : \ A\mathbf{x} = \mathbf{b}\}$$

defined for $\mathbf{b}$ in the nonnegative linear span of the columns of A.

- $\phi_A(\mathbf{b})$ counts the number of ways to express the vector $\mathbf{b}$ as a nonnegative integer linear combination of the columns of matrix A.

Remark: If

$$A = \left[\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array}\right],$$

we recover the **integer partition function**.

Classical Lie algebra of type A_r

$$\mathfrak{sl}_{r+1}(\mathbb{C}) = \{x \in M_{r+1}(\mathbb{C}) : \mathrm{Tr}(x) = 0\}$$

Definitions

Let e_i be the standard basis element of $\mathbb{R}^{r+1}$.

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \Leftarrow i^{th} place$$

Simple roots: $\alpha_i = e_i - e_{i+1}$.

Classical Lie algebra of type A_r

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Definitions

Simple roots: $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$

Positive roots: $\Phi^+ = \Delta \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i \leq j \leq r\}$

Negative roots: $\Phi^- = -\Phi^+$

Highest root: $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_r$

Definition

The **Weyl group** is a group generated by reflections, s_i, through hyperplanes that are orthogonal to the simple roots, α_i.

- It is isomorphic to the symmetric group on $r + 1$ letters, $\mathfrak{S}_{r+1}$.

Rules to Compute

$$s_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } i = j \\ \alpha_i + \alpha_j & \text{if } |i-j| = 1 \\ \alpha_j & \text{if } |i-j| > 1 \end{cases}$$

- Let's calculate $s_2 s_1(\alpha_1) = s_2(-\alpha_1) = -\alpha_1 - \alpha_2$.

Weyl group element σ	$\sigma(\alpha_1)$	$\sigma(\alpha_2)$	$\sigma(\alpha_1 + \alpha_2)$
1	α_1	α_2	$\alpha_1 + \alpha_2$
s_1	$-\alpha_1$	$\alpha_1 + \alpha_2$	α_2
s_2	$\alpha_1 + \alpha_2$	$-\alpha_2$	α_1
$s_1 s_2$	α_2	$-\alpha_1 - \alpha_2$	$-\alpha_1$
$s_2 s_1$	$-\alpha_1 - \alpha_2$	α_1	$-\alpha_2$
$s_1 s_2 s_1$	$-\alpha_2$	$-\alpha_1$	$-\alpha_1 - \alpha_2$

Table: Weyl group elements and their action on the roots of $\mathfrak{sl}_3(\mathbb{C})$

Kostant's Partition Function (KPF)

Let ξ be in the weight lattice and

$$\wp(\xi)$$

be the number of ways to write ξ as a nonnegative integral sum of positive roots.

Example

Calculate $\wp(\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6) = 4$.

$$\begin{array}{llll}
1\alpha_1 & +\,1\alpha_4 & +\,1\alpha_5 & +\,1\alpha_6 \\
1\alpha_1 & +\,1(\alpha_4 & +\,\alpha_5) & +\,1\alpha_6 \\
1\alpha_1 & +\,1\alpha_4 & +\,1(\alpha_5 & +\,\alpha_6) \\
1\alpha_1 & +\,1(\alpha_4 & +\,\alpha_5 & +\,\alpha_6)
\end{array}$$

Warning: Do not know of general formulas for the value of KPF.

Kostant's Weight Multiplicity Formula

The multiplicity of a weight μ in the irreducible representation of $\mathfrak{sl}_{r+1}(\mathbb{C})$ with highest weight λ can be computed via:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - \mu - \rho).$$

- $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$
- W is the Weyl group
- $\ell(\sigma)$ is the length of σ
- $\wp(\xi)$ is the KPF.

Note: $m(\lambda, \mu)$ is the dimension of the μ weight space.

Example

Type A_6, let $\mu = \alpha_2 + \alpha_3$, and observe

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6.$$

Calculate $m(\lambda, \mu) = m(\tilde{\alpha},\ \alpha_2 + \alpha_3)$.

Check the identity and simple reflections:

- $\sigma = 1$: $\wp(1(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6) = 4$
- $\sigma = s_1$: $\wp(s_1(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(-\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6) = 0$
- $\sigma = s_2$: $\wp(s_2(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 - \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6) = 0$
- $\sigma = s_3$: $\wp(s_3(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 - \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) = 0$
- $\sigma = s_4$: $\wp(s_4(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 + \alpha_5 + \alpha_6) = 2$
- $\sigma = s_5$: $\wp(s_5(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 + \alpha_4 + \alpha_6) = 1$
- $\sigma = s_6$: $\wp(s_6(\tilde{\alpha} + \rho) - \rho - \mu) = \wp(\alpha_1 + \alpha_4 + \alpha_5 - \alpha_6) = 0$

Thus, $m(\tilde{\alpha},\ \alpha_2 + \alpha_3) = 4 - 0 - 0 - 0 - 2 - 1 - 0 = 1$.

Question + Motivation

$\boxed{\text{What elements of the Weyl group contribute nontrivially to KWMF?}}$

Definition

For λ, μ integral weights of $\mathfrak{g}$, the **Weyl alternation set** is

$$\mathcal{A}(\lambda, \mu) = \{\sigma \in W \ : \ \wp(\sigma(\lambda + \rho) - \mu - \rho) > 0\}.$$

Example (Cont.)

$$\mathcal{A}(\tilde{\alpha}, \alpha_2 + \alpha_3) = \{1,\ s_4,\ s_5\}$$

In A_6, there are $7! = 5040$ elements in W, but only three contribute to $m(\tilde{\alpha},\ \alpha_2 + \alpha_3)$.

Weyl Alternation Sets

Definition

For λ, μ integral weights of $\mathfrak{g}$, the **Weyl alternation set** is

$$\mathcal{A}(\lambda,\mu) = \{\sigma \in W \ : \ \wp(\sigma(\lambda+\rho) - \mu - \rho) > 0\}.$$

Some known results:

1. In type A_r, if λ is the highest root and $\mu = 0$, then $|\mathcal{A}(\lambda, 0)| = F_{r+1}$ (P.E. Harris, 2012).
2. In type A_2, if λ is a dominant integral weight and μ is in the root lattice, then $\mathcal{A}(\lambda,\mu)$ is known (P.E. Harris, G. Mabie, H. Lencisky, 2017).
3. In types B_r, C_r, D_r, $\mathcal{A}(\lambda, 0)$ is known when λ is a sum of all simple roots (P.E. Harris, K. Cheng, E. Insko, 2020).
4. In type A_r, if λ is the highest root and μ is a positive root, then $|\mathcal{A}(\lambda,\mu)| = F_i \cdot F_{r-j+1}$ (K. J. Harry, 2023).

Positive Roots $\to$ Negative Roots

Specifically, let $\mu \in \Phi^- = -\Phi^+$.

Questions

- What elements are in $\mathcal{A}(\lambda, \mu)$?
- How many elements are in $\mathcal{A}(\lambda, \mu)$?

Note: We specialize to $\lambda = \tilde{\alpha}$ and $\mu = -\tilde{\alpha}$.

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Moving on...

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Moving on...
Idea: A subword is like a letter in a new alphabet.

Forbidden Subwords

Lemma (MRC 2024, Anderson, et al.)

The words

$$s_2 s_1, \qquad s_1 s_2, \qquad s_{r-1} s_r, \qquad \textit{and} \qquad s_r s_{r-1},$$

and for any $2 \leq i \leq r-1$*, the words*

$$s_{i-1} s_i s_{i+1}, \qquad s_i s_{i-1} s_{i+1}, \qquad \textit{and} \qquad s_{i+1} s_i s_{i-1}$$

are not contained in $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$*.*

Lemma (MRC 2024, Anderson, et al.)

Let $k \in [r-3]$*. The product of the four simple reflections* s_k*,* s_{k+1}*,* s_{k+2}*, and* s_{k+3} *in any order is not contained in* $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$*.*

Basic Allowable Subwords

Let $\mathsf{BAS}(\lambda, \mu)$ denote the set of basic allowable subwords corresponding to the pair λ and μ.

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Proposition (MRC 2024, Anderson, et al.)

If σ is of the form

(a) s_k *with* $1 \leq k \leq r$,

(b) $s_{k+1}s_k$ *with* $2 \leq k \leq r-2$,

(c) $s_k s_{k+1}$ *with* $2 \leq k \leq r-2$,

(d) $s_k s_{k+1} s_k$ *with* $2 \leq k \leq r-2$, *or*

(e) $s_{k+2}s_k s_{k+1}$ *with* $2 \leq k \leq r-3$,

then $\sigma \in \mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$.

Key Insight.

In all cases, $\sigma(\tilde{\alpha}+\rho)+\tilde{\alpha}-\rho = 2\tilde{\alpha}-\sum_{i=1}^{r} c_i\alpha_i$ with all $c_i \leq 2$. $\square$

Basic Allowable Subwords

Let $\mathrm{BAS}(\lambda, \mu)$ denote the set of basic allowable subwords corresponding to the pair λ and μ.

Theorem (MRC 2024, Anderson, et al.)

The set $\mathrm{BAS}(\tilde{\alpha}, -\tilde{\alpha})$ *of* $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$ *consists of*

(a) s_k *with* $1 \leq k \leq r$,

(b) $s_{k+1}s_k$ *with* $2 \leq k \leq r-2$,

(c) $s_k s_{k+1}$ *with* $2 \leq k \leq r-2$,

(d) $s_k s_{k+1} s_k$ *with* $2 \leq k \leq r-2$, *and*

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BAS Example

Example

	in general	in Type A_6
(a)	s_k	$s_1, s_2, s_3, s_4, s_5, s_6$
(b)	$s_{k+1}s_k$	s_3s_2, s_4s_3, s_5s_4
(c)	s_ks_{k+1}	s_2s_3, s_3s_4, s_4s_5
(d)	$s_ks_{k+1}s_k$	$s_2s_3s_2, s_3s_4s_3, s_4s_5s_4$
(e)	$s_{k+2}s_ks_{k+1}$	$s_4s_2s_3, s_5s_3s_4$

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(d)	$s_ks_{k+1}s_k$	$s_2s_3s_2, s_3s_4s_3, s_4s_5s_4$
(e)	$s_{k+2}s_ks_{k+1}$	$s_4s_2s_3, s_5s_3s_4$

$BAS(\tilde{\alpha}, -\tilde{\alpha})$:

- contains $s_2s_3s_2 \cdot s_5$ (using (d) and (a)),

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(c)	s_ks_{k+1}	s_2s_3, s_3s_4, s_4s_5
(d)	$s_ks_{k+1}s_k$	$s_2s_3s_2, s_3s_4s_3, s_4s_5s_4$
(e)	$s_{k+2}s_ks_{k+1}$	$s_4s_2s_3, s_5s_3s_4$

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- contains $s_2s_3s_2 \cdot s_5$ (using (d) and (a)),
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(c)	s_ks_{k+1}	s_2s_3, s_3s_4, s_4s_5
(d)	$s_ks_{k+1}s_k$	$s_2s_3s_2, s_3s_4s_3, s_4s_5s_4$
(e)	$s_{k+2}s_ks_{k+1}$	$s_4s_2s_3, s_5s_3s_4$

$BAS(\tilde{\alpha}, -\tilde{\alpha})$:

- contains $s_2s_3s_2 \cdot s_5$ (using (d) and (a)),
- does not contain $s_1s_2s_3$ (since s_1s_2 is a forbidden subword), and
- does not contain $s_5 \cdot s_2 \cdot s_3 \cdot s_4$ (as a product of four simple reflections).

Moving On

We also have analogous characterizations for when μ is not the negative highest root, i.e. when

- $\mu = -(\alpha_1 + \alpha_2 + \cdots \alpha_j)$ for $1 \leq j < r$,
- $\mu = -(\alpha_i + \alpha_{i+1} + \ldots + \alpha_r)$ for $1 < i \leq r$, and
- $\mu = -(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j)$ for $1 < i \leq j < r$.

Theorem (Harry, 2023)

Fix $1 \leq i \leq j \leq r$ and let $\tilde{\alpha}$ be the highest root of $\mathfrak{sl}_{r+1}(\mathbb{C})$. Then,

$$|\mathcal{A}(\tilde{\alpha}, \alpha_i + \alpha_{i+1} + \cdots + \alpha_j)| = F_i \cdot F_{r-j+1}$$

where F_n denotes the n-th Fibonacci number.

Enumeration

Proposition (MRC 2024, Anderson, et al.)

Let $r \geq 1$ and fix $1 \leq i \leq r$.

1. *If $i = 1$ or $i = r$, then $|\mathcal{A}_r(\tilde{\alpha}, -\alpha_i)| = F_{r+1}$.*
2. *If $r > 2$ and $2 \leq i \leq r-1$, then* $|\mathcal{A}_r(\tilde{\alpha}, -\alpha_i)| = F_r + F_{i-1}F_{r-i-1} + F_{i-2}F_{r-i}$.

Lemma (MRC 2024, Anderson, et al.)

The number of subsets of $[n]$ that do not contain a pair of consecutive numbers is F_{n+2}.

Proof Sketch (of 1.).

Notice $\mathcal{A}_r(\tilde{\alpha}, -\alpha_1)$ consists of commuting products of $r-1$ simple transpositions. $\square$

A q-analog

Recall Lusztig's definition of the q-analog of Kostant's partition function defined (1983), as the polynomial-valued function

$$\wp_q(\xi) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i equals the number of ways to write ξ as a sum of exactly i positive roots.

Then, the q-analog of Kostant's weight multiplicity formula is

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - \mu - \rho).$$

Future Work

In type A_r, using $\mathcal{A}(\tilde{\alpha}, 0)$, Harris shows (combinatorially)

$$m_q(\tilde{\alpha}, 0) = \sum_{1 \leq i \leq r} q^i.$$

In type A_r, using $\mathcal{A}(\tilde{\alpha}, \mu)$ with $\mu \in \Phi^+$, Harry shows that $m_q(\tilde{\alpha}, \mu)$ is a power of q. Harry conjectured the following:

Conjecture (Harry, 2023)

If $\tilde{\alpha}$ is the highest root and $\mu = -\alpha_{i,j}$ with $1 \leq i \leq j \leq r$ is a negative root of the Lie algebra of type A_r, then

$$m_q(\tilde{\alpha}, \mu) = q^{r+j-i+1} + q^{r+j-i} - q^{j-i+1}.$$

We have a proof for when $\mu = -\alpha_i$ for $1 \leq i \leq r$.
More to be done!

Thank you!

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Contact Email: kph3@iastate.edu

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