c-singleton Birkhoff polytopes and order polytopes of heaps

Esther Banaian (UC Riverside)
j. with Sunita Chepuri, Emily Gunawan, and Jianping Pan

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Motivating Question

The following sequence (OEIS: A003121)

$$1, 1, 1, 2, 12, 286, 33592, 23178480, \ldots$$

is known to count

- shifted standard Young tableaux of staircase shape,
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Is this pattern-avoiding Birkhoff polytope unimodularly equivalent to an order polytope?

Theorem(BCGP)

Yes+

c-singletons

We will work in S_{n+1} throughout, and we will fix a *Coxeter element* $c = a_1 \cdots a_n$ (i.e. $\{a_1, \ldots, a_n\} = \{s_1, \ldots, s_n\}$).

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- Let c^∞ be the infinite word $a_1 \dots a_n | a_1 \dots a_n | \cdots$.
- Given $w \in S_{n+1}$, let $\mathrm{sort}_c(w)$ be the reduced word of w which is lexicographically first as a subword of c^∞.

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- **Example:** $c = s_1 s_3 s_2$, $c^\infty = s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 \cdots$.
  If $w = s_1 s_2 s_1 = s_2 s_1 s_2$, $\mathrm{sort_c}(w) = s_1 s_2 s_1$.

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Definition/Theorem (Hohlweg-Lange-Thomas)

$w \in S_{n+1}$ is a *c-singleton* if and only if w has a reduced word which is a prefix of a reduced word in the commutation class of $\text{sort}_c(w_0)$.

Example If $c = s_1 s_3 s_2$, $\text{sort}_c(w_0) = s_1 s_3 s_2 s_1 s_3 s_2$, and $w = s_2 s_1 s_2$ is not a c-singleton.

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c-singletons are a subset of c-sortable elements, which are the elements of Reading's c-Cambrian lattices. The c-singletons also form a distributive sublattice of weak order.

Pattern-avoidance classification

Given c, for any $i \in [2, n]$, let $i \in \underline{[2, n]}$ if s_i appears after s_{i-1} in c and otherwise $i \in \overline{[2, n]}$.

Example: Let $c = s_4 s_2 s_5 s_1 s_3 \in S_6$. Then, $\underline{[2, 5]} = \{3, 5\}$ and $\overline{[2, 5]} = \{2, 4\}$.

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Say that w avoids pattern $13\underline{2}$ if w does not contain the pattern 132 with "2" lower-barred - that is, we do *not* have $w = ...x...y...z..$ with $x < z < y$ and $z \in \underline{[2,n]}$.

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Proposition[Reading]

A permutation w is a c-singleton if w a avoids $13\underline{2}, 31\underline{2}$, $\overline{2}31$, and $\overline{2}13$.

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Example: If $c = s_1 s_3 s_2$, $\underline{[2, 3]} = \{2\}$ and $\overline{[2, 3]} = \{3\}$. We observe 4213 is c-singleton while $\mathbf{\overline{3}24}1$ is not.

c-Birkhoff

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Example: If $c = s_1 s_3 s_2$, $\mathrm{Birk}(c)$ is the convex hull of the following 9 points in $\mathbb{R}^{16}$

$$\underset{\mathrm{Id}}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}} \quad \underset{s_1}{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}} \quad \underset{s_3}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}} \quad \underset{s_1 s_3}{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}} \quad \underset{s_1 s_3 s_2}{\begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}}$$

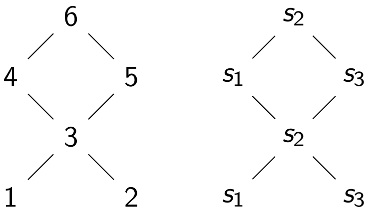
$$\underset{s_1 s_3 s_2 s_1}{\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix}} \quad \underset{s_1 s_3 s_2 s_3}{\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}} \quad \underset{s_1 s_3 s_2 s_1 s_3}{\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}} \quad \underset{s_1 s_3 s_2 s_1 s_3 s_2}{\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}}$$

Heaps

Definition

Let $[u] = u_1 \cdots u_\ell$ be a reduced word. The *heap* from $[u]$ is the the poset on $\{1, \ldots, \ell\}$ which is the transitive closure of relations $x \prec y$ whenever $x < y$ and u_x and u_y do not commute.

Example Given $w = s_1 s_3 s_2 s_1 s_3 s_2 = u_1 u_2 u_3 u_4 u_5 u_6$, we draw (1) Hasse diagram of the heap and (2) the same diagram but with u_i in place of i.

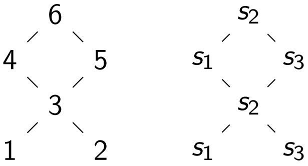


Order ideals and linear extensions of a heap

A *linear extension* of a poset is a total order on the set which respects the partial order. For example, there are 4 linear extensions of the poset below.

Proposition [Stembridge]

Linear extensions of a Heap($[u]$) are in bijection with the commutation class of $[u]$.



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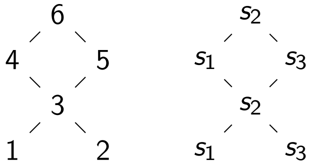
Proposition [Stembridge]

Linear extensions of a Heap($[u]$) are in bijection with the commutation class of $[u]$.

A *order ideal* of a poset $(P, \preceq)$ is a set $I \subseteq P$ such that if $x \in I$ and $y \preceq x$, $y \in I$.

Rephrasing (Hohlweg-Lange-Thomas)

Order ideals of Heap($\mathrm{sort}_c(w_0)$) are in bijection with c-singletons.



Order Polytopes

Order Polytope [Stanley]

Given a poset P, the *order polytope* of P, $\mathcal{O}(P)$, lives in $\mathbb{R}^P$ and is the convex hull of indicator vectors of the order ideals of P.

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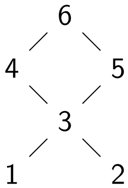

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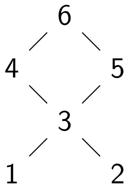

Vertices of $\mathcal{O}(P)$

1. $(0,0,0,0,0,0)$
2. $(1,0,0,0,0,0)$
3. $(0,1,0,0,0,0)$
4. $(1,1,0,0,0,0)$
5. $(1,1,1,0,0,0)$
6. etc (4 more)

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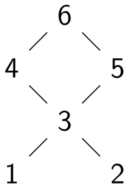

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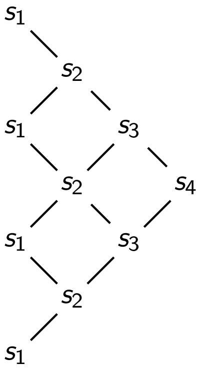
Theorem[Stanley]

The normalized volume of $\mathcal{O}(P)$ is equal to the number of linear extensions of P.

Let $c = s_1 s_2 \cdots s_n$, so $\underline{[2,n]} = [2,n]$, $\overline{[2,n]} = \emptyset$.

- The c-singletons are 312 and 132 avoiding permutations. There are 2^n.
- The c-Cambrian lattice is the Tamari lattice.

The heap of $\mathrm{sort}_c(w_0)$ has the following form:



Return to Motivation

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The following sequences are equal:

- number of shifted standard Young tableaux of staircase shape
- number of longest chains in the Tamari lattice
- normalized volume of Birk(c) (Davis-Sagan)

Question (Davis-Sagan)

For $c = s_1 s_2 \cdots s_n$, is Birk(c) unimodularly equivalent to an order polytope?

Main Result: Yes+

Theorem (BCGP)

There is a unimodular equivalence between $\mathrm{Birk}(c)$ *and* $\mathcal{O}(\mathrm{Heap}(\mathrm{sort}_c(w_0)))$.

When $c = s_1 s_2 \cdots s_n$, this answers **yes** to Davis and Sagan's question.

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Proof ideas:

1. Notice $\mathrm{Birk}(c)$ is in $(n+1)^2$-space while $\mathcal{O}(\mathrm{Heap}(\mathrm{sort}_c(w_0)))$is in $\binom{n+1}{2}$-space.
2. We use Reading's pattern avoidance criteria to describe a projection of $\mathrm{Birk}(c)$.

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2. We use Reading's pattern avoidance criteria to describe a projection of $\mathrm{Birk}(c)$.
3. Then we the existence of a unimodular transformation from the projection of $\mathrm{Birk}(c)$ to $\mathcal{O}(\mathrm{Heap}(\mathrm{sort}_c(w_0)))$.

Zero Relations

To give these projections, we study relations on the affine space spanned by $\{X_w : w \text{ is a } c\text{-singleton}\}$: aff$(c)$.
For example, let $c = [1432657]$. The following entries are 0 for every point in aff(c).

		$\overline{3}$	$\overline{4}$		$\overline{6}$		
		X	X		X		
		X	X		X		
			X				
				X			
				X			
	X			X		X	
	$\underline{2}$			$\underline{5}$		$\underline{7}$	

Summing Relations

For every point in aff([1432657]), the sum of the entries in like-suited boxes is the same.

0	1	−1	−2	2	5	3	4
♣	♡	X	X	♠	X	♣	♡
♣	♡	X	X	♠	X	♣	♡
♣	♡	♠	X	♠	♠	♣	♡
				X			
				X			
	X			X		X	

Tamari Case

If $c = [123 \ldots n]$, then the relations are simpler. The zero relations are:

			X	X			
		X	X	X	X		
	X	X	X	X	X	X	

Corollary

Given a 132 and 312 avoiding permutation w, for any $1 \leq m \leq n+1$, the values $w(1), w(2), \ldots, w(m)$ are all distinct modulo m.

(This is equivalent to zero *and* summing relations for $c = [123 \ldots n]$)

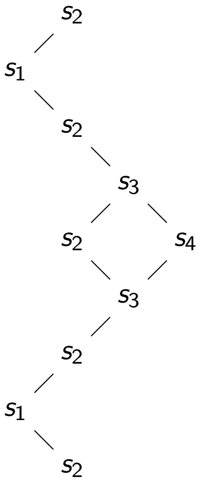
Future Work

Given any reduced word and heap $H([u])$, one can define $\mathrm{Birk}([u])$ to be the convex hull of permutation matrices arising from linear extensions of u.

Question

When is $\mathcal{O}(H([u]))$ unimodularly equivalent to $\mathrm{Birk}([u])$?

The answer will depend on $[u]$ and not just the corresponding element of S_{n+1}. For $[u] = [2123243212]$, $\mathrm{Birk}([u])$ is 9-dimensional while $\mathcal{O}(H([u]))$ is 10-dimensional.



Thank you

Thank you for listening!