Supersolvable convex geometries and the flag vectors associated to Hopf monoids

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Hopf monoid

A **Hopf monoid** (Aguiar-Mahajan) H consists of:

- One vector space $\mathrm{H}[I]$ for each finite set I.
- Two linear maps

$$\mu_{S,T} : \mathrm{H}[S] \otimes \mathrm{H}[T] \to \mathrm{H}[I] \quad \text{and} \quad \Delta_{S,T} : \mathrm{H}[I] \to \mathrm{H}[S] \otimes \mathrm{H}[T]$$

  for each finite set I and each decomposition $I = S \sqcup T$.

These are subject to certain simple axioms. We assume that $\mathrm{H}[\emptyset] = \mathbb{k}$.

Characters

A **character** ψ on a Hopf monoid H is a collection of linear maps

$$\psi_I : \mathrm{H}[I] \to \Bbbk$$

for each finite set I subject to certain simple axioms. *The set of all characters on* H *form a group*. Denote the group operation $*$.

Define another character $\overline{\psi}$ by

$$\overline{\psi}_I(x) = (-1)^{|I|}\psi_I(x).$$

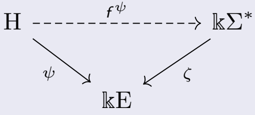
Definition

ψ is **odd** if ψ and $\overline{\psi}$ are inverse in the character group.

Flag f-vectors

Theorem (Aguiar-Bergeron-Sottile, Aguiar-Mahajan)

For any Hopf monoid vector species H *with a character* $\psi : \mathrm{H} \to \Bbbk\mathrm{E}$, *there exists a unique morphism of Hopf monoids* $f : \mathrm{H} \to \Bbbk\Sigma^*$ *(the Hopf monoid of compositions) such that the following diagram commutes.*



$$f_I^{\psi}(x) = \sum_{F \vDash I} \psi_F \Delta_F(x) \mathtt{M}_F. \text{ Denote } f_F^{\psi}(x) = \psi_F \Delta_F(x).$$

The *flag f-vector* associated with ψ is the vector $(f_\alpha^{\psi}(x))_\alpha$, $\alpha \vDash |I|$,

$$f_\alpha^{\psi}(x) = \sum_{F:\, \mathrm{type}(F) = \alpha} f_F^{\psi}(x).$$

Flag h-vector, ab and cd-indices associated to a character

The flag h-vector for ψ is determined by the following relation.

$$h_\alpha^\psi(x) = \sum_{\alpha' \leq \alpha} (-1)^{l(\alpha)-l(\alpha')} f_{\alpha'}^\psi(x).$$

For $\alpha = (\alpha_1, ..., \alpha_k) \vDash n$, let $m(a,b)_\alpha$ denote the ab-monomial of degree $n-1$ with b's on position $\alpha_1, \alpha_1 + \alpha_2, ..., \alpha_1 + ... + \alpha_{k-1}$.
Example: If $\alpha = (2,1,1,5,3) \vDash 12$ then $m(a,b)_\alpha = abbbaaaabaa$.

The *ab-index* associated with ψ of $x \in H[I]$ is

$$\Psi_x^\psi(a,b) = \sum_{\alpha \vDash |I|} h_\alpha^\psi(x) m(a,b)_\alpha.$$

Let $c = a+b$, $d = ab+ba$. The *cd-index*, if it exists, is the polynomial $\Phi(c,d)$ such that $\Phi(c,d) = \Psi(a,b)$.

Existence of *cd*-indices for odd Characters

Theorem (Aguiar-Bergeron-Sottile)

If the character ψ is odd, then for any $x \in \mathrm{H}[I]$ the vector $(f_\alpha^\psi(x))_\alpha$ satisfies the **Bayer-Billera** *(or* **generalized Dehn-Sommerville***) relations.*

Proposition (Fine, Bayer - Klapper)

The cd-index of x exists if and only if the vector $(f_\alpha^\psi(x))_\alpha$ satisfies the Bayer-Billera relations.

Convex Geometries

Let 2^I denote the set of subsets of a finite set I.

Definition

A **closure operator** on I is a map $c : 2^I \to 2^I$ such that for all $A, B \in 2^I$,

- $A \subseteq c(A)$,
- if $A \subseteq c(B)$, then $c(A) \subseteq c(B)$.

A subset $K \subseteq I$ is **closed** if $c(K) = K$.

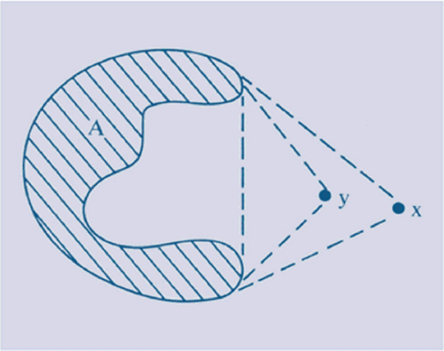
A **convex geometry** with ground set I is a closure operator g on I that satisfies the **antiexchange axiom**:

- if $a \in g(A \cup \{b\})$, $a \notin g(A)$, and $a \neq b$, then $b \notin g(A \cup \{a\})$.

A closed set of g is said to be **convex**.

We assume from now on that all convex geometries g are **loopless**: $g(\emptyset) = \emptyset$.

Convex Geometries



From the cover of *Matroid Applications*, edited by Neil White.

Operations of convex geometries

Consider $I = S \sqcup T$. Let c_1, c_2 be two convex geometries on ground set S, T, respectively, the **direct sum** $c_1 \oplus c_2$ is a convex geometry defined on I with

$$(c_1 \oplus c_2)(A) = c_1(A \cap S) \cup c_2(A \cap T).$$

Let c be a convex geometry on ground set I, then the **restriction** of c on S, namely $c|_S$, and the **contraction** of c by S, namely $c/_S$, are convex geometries on ground sets S, T respectively, defined as follows.

$$(c|_S)(B) = c(B) \cap S, \ (c/_S)(C) = c(S \cup C) \cap T.$$

Hopf monoid of convex geometries

Let $\mathrm{CG}[I]$ be the vector space spanned by all convex geometries on I. Let

- $\mu_{S,T}(c, d) = c \oplus d$,
- $\Delta_{S,T}(e) = \begin{cases} e|_S \otimes e/_S & \text{if } S \text{ is convex,} \\ 0 & \text{otherwise.} \end{cases}$

Then CG is a Hopf monoid.

Canonical characters of convex geometries

Set the canonical character $\eta_I(g) = 1$. Let

$$\zeta = \overline{\eta}^{-1}, \qquad \varphi = \zeta * \eta, \qquad \varphi' = \eta * \zeta.$$

Proposition (M)

- $\zeta_I(g) = \begin{cases} 1 & \textit{if } g(A) = A \textit{ for all } A \subseteq I, \\ 0 & \textit{otherwise.} \end{cases}$
- *$\varphi_I(g) = (\zeta * \eta)_I(g)$ counts $K \subseteq I$ such that for all $K' \subseteq K$, K' convex.*
- *$\varphi'_I(g) = (\eta * \zeta)_I(g)$ counts $K \subseteq I$ such that for all $x \in K$, $I \backslash x$ is convex.*

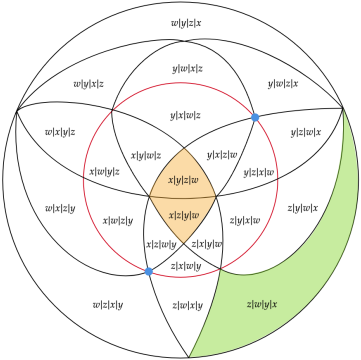
Note φ, φ' are odd characters.

Supersolvable convex geometries

Definition

- (Stanley) A lattice L is *supersolvable* if it admits a maximal chain $\mathfrak{c}$ such that for all chain m in L, the smallest sublattice in L containing $\mathfrak{c}$ and m is distributive. We call $\mathfrak{c}$ a *chief chain* of L.
- (Armstrong) A convex geometry g is supersolvable if L_g, the lattice of convex sets, is supersolvable.

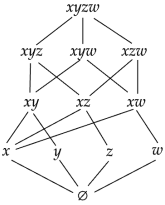
The braid arrangement of $\{x, y, z, w\}$



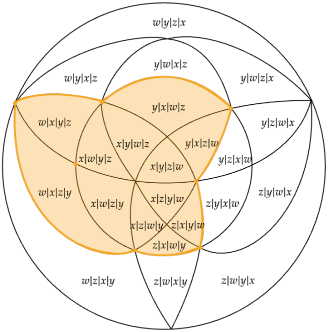
Some bijections between combinatorial and geometric descriptions:

face —— set composition
chamber —— linear order
top cone —— partial order

Example: order complex of a supersolvable convex geometry



L_g



$\Delta\left(\overline{L(g)}\right)$

Geometric Characterization of Supersolvable Convex Geometries

Theorem (M)

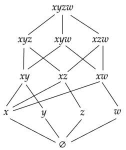
Let g be a convex geometry. Let $V_g = \Delta(\overline{L(g)})$. Let $V_{p_1},\ldots, V_{p_k}$ be the maximal top cones in V_g (they correspond to partial orders $p_1,\ldots, p_k$, respectively). Then the following statements are equivalent.

(1) $\bigcap_{i=1}^{k} V_{p_i}$ *contains at least one chamber.*

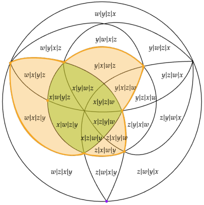
(2) *There exists a partial order p_0 on I with $V_{p_0} = \bigcap_{i=1}^{k} V_{p_i}$.*

(3) *g is supersolvable.*

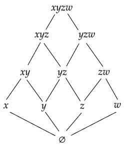
Example and Non-example



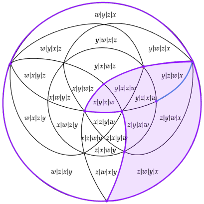
g_1 (supersolvable)



$\Delta(\overline{L(g_1)})$



g_2 (not supersolvable)



$\Delta(\overline{L(g_2)})$

ab-indices

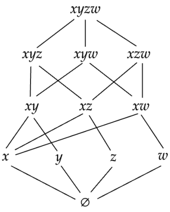
Let g be a supersolvable convex geometry, and let p_0 be the partial order with V_{p_0} the intersection of all maximal top cones V_p in V_g. Let $\overline{p_0}$ be the partial order obtained by reversing relations in p_0. Fix any linear order ℓ_0 satisfying $\ell_0 \in V_{p_0}$ (so $\overline{\ell_0} \in V_{\overline{p_0}}$,).

Theorem (M)

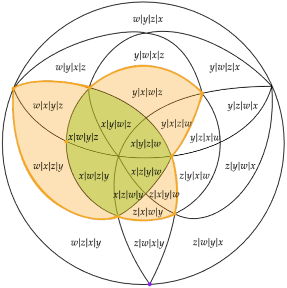
Let $\alpha \vDash n$.

- $[m(a,b)_\alpha]\Psi_g^\eta = |\{\ell \in V_g \mid \mathrm{Des}\left(\begin{pmatrix} \ell_0 \\ \ell \end{pmatrix}\right) = \alpha\}|$.
- $[m(a,b)_\alpha]\Psi_g^\zeta = |\{\ell \in V_g \mid \mathrm{Des}\left(\begin{pmatrix} \overline{\ell_0} \\ \ell \end{pmatrix}\right) = \alpha\}|$.

Example (for ζ)



g_1 (supersolvable)



$\Delta(\overline{L(g_1)})$

For g_1, V_{p_0} is in green. Let $\ell_0 = x|y|z|w$, so $\overline{\ell_0} = w|z|y|x \in V_{\overline{p_0}}$. Then $\Psi_{\zeta,g_1}(a,b) = baa + 3aba + 2bba + 2bab + 3abb + b^3$.

- baa corresponds to the chamber $x|w|z|y$.
- $3aba$ corresponds to the chambers $w|x|y|z$, $y|x|w|z$, $z|x|w|y$, etc.
- we can obtain the coefficients of the remaining ab-monomials by the same procedure.

cd-index

There is bijective correspondence between the set of non- commutative cd-monomials of degree $n-1$ and the set of sparse subsets of $[n-1]$.

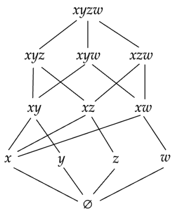
$$c^{a_1}dc^{a_2}d\ldots c^{a_k}dc^{a_{k+1}} \leftrightarrow \{\deg(c^{a_1}d), \deg(c^{a_1}dc^{a_2}d), \ldots, \deg(c^{a_1}d\ldots c^{a_k}d)\}.$$

Example: $cdcdd \longleftrightarrow \{3,6,8\}$.
Given a sparse subset S, let $m(c,d)_S$ be the corresponding cd-monomial.
Let g be a supersolvable convex geometry with p_0 satisfying V_{p_0} is the intersection of all maximal top cones V_p in V_g. Fix any linear order ℓ_0 satisfying $\ell_0 \in V_{p_0}$ (so $\overline{\ell_0} \in V_{\overline{p_0}}$).

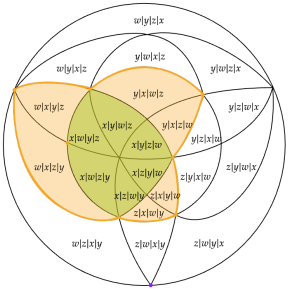
Theorem (M)

- $[m(c,d)_S]\Phi_g^{\varphi'} = 2^{|S|+1} \cdot |\{\ell \in V_g \mid \mathrm{Peak}(\begin{pmatrix} \overline{\ell_0} \\ \ell \end{pmatrix}) = S\}|.$
- $[m(c,d)_S]\Phi_g^{\varphi} = 2^{|S|+1} \cdot |\{\ell \in V_g \mid \mathrm{Peak}(\begin{pmatrix} \ell_0 \\ \ell \end{pmatrix}) = S\}|.$

Example (for φ')



g_1 (supersolvable)



$\Delta(\overline{L(g_1)})$

For g_1, we have $\Phi_{\varphi',g_1}(c,d) = 8c^3 + 8cd + 24dc$. Fix $\ell_0 = x|y|z|w$ so $\overline{\ell_0} = w|z|y|x \in V_{\overline{p_0}}$. Then we have $8c^3$ comes from 4 chambers $x|w|z|y$, $x|y|z|w$, $x|z|w|y$, $x|y|w|z$ with no peaks with respect to $\overline{\ell_0}$. $8cd$ comes from 2 chambers $x|w|y|z$, $x|z|y|w$, each of which has one peak on position 3 with respect to $\overline{\ell_0}$. $24dc$ comes from 6 chambers $w|x|y|z$, $w|x|z|y$, $y|x|w|z$, $y|x|z|w$, $z|x|w|y$, $z|x|y|w$, each of which has one peak on position 2 with respect to $\overline{\ell_0}$.

Thank you!