On Computational Research Methods

Jennifer Elder

AMS Spring Central Sectional Meeting
March 29, 2025

Outline

Brief outlines of two projects, focused on the following:

1. How computational tools and databases can help new comers jump into a research area.
2. How definitions in mathematical programs and experiments without any particular goal in mind can lead to interesting combinatorial results.
3. How under-utilized FindStat is as starting point for research questions.

First Project

Motivating Question: How many Stirling permutations are flattened?

Collaborators (2023)

Azia Barner

Adam Buck

Kim Harry

Pamela E. Harris

Anthony Simpson

Mathematica Flattened function

Given $\mathcal{B} = \{b_1, b_2, \ldots, b_k\}$ a set partition of $[n]$, the Mathematica function

$$\texttt{Flatten}(\mathcal{B})$$

returns the permutation of $[n]$ arising from simply erasing the block dividers.

However, the output is dependent on the order of the blocks and the order of the elements in the blocks.

Example

If

$$\mathcal{B} = \{\{9\}, \{2, 3, 5, 8\}, \{1, 10\}, \{4, 6, 7\}\},$$

then

$$\texttt{Flatten}(\mathcal{B}) = 923581(10)467.$$

Callan - formalized flattened definition

The *standard increasing* form of a set partition is the order of the blocks $\mathcal{B} = b_1 \mid b_2 \mid \cdots \mid b_k$ such that:

1. elements in blocks are ordered from smallest to largest,
2. blocks are ordered by their minimal elements.

Callan - formalized flattened definition

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Example (Continued.)

Ordering the blocks of $\mathcal{B}$ and its elements as above yields

$$\underbrace{1, 10}_{b_1} \mid \underbrace{2, 3, 5, 8}_{b_2} \mid \underbrace{4, 6, 7}_{b_3} \mid \underbrace{9}_{b_4}.$$

Then

$$\texttt{Flatten}(\mathcal{B}) = 1(10)23584679.$$

Flattened words of length n

Definition

A word $w \in [n]^n$ is a *weakly flattened word with runs of weak ascents* if the leading values of its runs appear in weakly increasing order.

Flattened words of length $n \subseteq$ Words of length n

Example

Words	Leading Terms	Flattened?
$\underline{1}$ 1 2 5 $\underline{1}$ 4 $\underline{3}$	$1 \leq 1 \leq 3$	✓
$\underline{1}$ 1 2 5 $\underline{4}$ $\underline{1}$ 3	$1 \leq 4 \not\leq 1$	$\times$

Stirling Permutations

A **Stirling permutation** of order n is a permutation on the multiset $[n]_2 = \{1, 1, 2, 2, \ldots, n, n\}$ such that

- numbers between the i values must be greater than i,

we refer to such values as being *nested* between i.

Example

$$123231 \notin \mathcal{Q}_3$$

1. Note that the value 2 lies between the 3's and so this is not a Stirling permutation.

Flattened Stirling Permutations

Example

If $w = \underbrace{1233}_{\sigma_1}\ \underbrace{2}_{\sigma_2}\ \underbrace{1}_{\sigma_3} \in \mathcal{Q}_3$

then

$$1 \ \leq \ 2 \not\leq 1.$$

Thus, $123321 \notin \text{flat}(\mathcal{Q}_3)$.

Flattened Stirling Permutations

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If $w = \underbrace{1233}_{\sigma_1}\ \underbrace{2}_{\sigma_2}\ \underbrace{1}_{\sigma_3} \in \mathcal{Q}_3$

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Example

If $w = \underbrace{122}_{\sigma_1}\ \underbrace{133}_{\sigma_2} \in \mathcal{Q}_3$

then

$$1 \ \leq \ 1.$$

Thus, $122133 \in \text{flat}(\mathcal{Q}_3)$.

Research Question

Computationally, we found:

n	1	2	3	4	5	6	7	8	9	10
$\lvert\mathrm{flat}(\mathcal{Q}_n)\rvert$	1	2	6	24	116	648	4088	28640	219920	1832224

Table: Number of flattened Stirling permutations

Note that the cardinalities of flat($\mathcal{Q}_n$) are identically to the Dowling numbers, described in OEIS A007405.

"This is the number of type B set partitions, see R. Suter."
- Per W. Alexandersson, Dec 19. 2022

Research Question

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New Goal: Give a bijection between flattened Stirling permutations of order n and type B set partitions.

Example for Type B Set Partition

Type B: set partitions on the interval of integers $[-n, n]$.

Example

Note that for $n = 5$,

$$\pi = \{\{0, 1, -1, 2, -2\}, \{3, -4\}, \{-3, 4\}, \{5\}, \{-5\}\}$$

is a type B set partition of $[-5, 5]$.

- Note π is a set partition.
- For $\beta = \{3, -4\} \in \pi$, then $-\beta = \{-3, 4\} \in \pi$.
- For $\beta = \{5\} \in \pi$, then $-\beta = \{-5\} \in \pi$.
- Zero-block: $\beta = \{0, 1, -1, 2, -2\}$, and $\beta = -\beta$.

Our Main Result

Theorem

For $n \geq 1$, the set Π^B_{n-1} is in bijection with the set $\text{flat}(\mathcal{Q}_n)$.

Sketch of Proof: ($\Rightarrow$) Given a type B set partition,

$$\pi = \left\{ \begin{matrix} \{0\}, \{1\}, \{-1\}, \{4\}, \{-4\}, \{2,7,-8\}, \{-2,-7,8\}, \\ \{3,5,6,-9,-10\}, \{-3,-5,-6,9,10\} \end{matrix} \right\}$$

- Adler's notation and bump up each of the number's magnitude,

$$1 \mid 2 \mid \overline{9}38 \mid (\overline{10})(\overline{11})467 \mid 5$$

- duplicate each number and nest accordingly,

$$1\ 1 \mid 2\ 2 \mid \overline{9}\ \overline{9}\ 3\ 8\ 8\ 3 \mid (\overline{10})\ (\overline{10})\ (\overline{11})\ (\overline{11})\ 4\ 6\ 6\ 7\ 7\ 4 \mid 5\ 5$$

- and eliminate the bars and dividers.

$$\underbrace{1\ 1\ 2\ 2\ 9\ 9}\ \underbrace{3\ 8\ 8}\ \underbrace{3\ 10\ 10\ 11\ 11}\ \underbrace{4\ 6\ 6\ 7\ 7}\ \underbrace{4\ 5\ 5}$$

Second Project

Motivating Question: Using a database of permtuation statisitics, how many instances of the homomesy phenomenon exist?

Collaborators (2021)

Joint work with Jessica Striker, Erin McNicholas, Nadia Lafrenière, and Amanda Welch

A Large Database Study

- Consider a set, an action on that set, and a statistic.
- Homomesy: whenever the average of the statistic over the whole set is the same as the average of the statistic over every orbit.
- Our set was S_n, and the actions on S_n and statistics on permutations came from considering all possible combinations found in the FindStat Database.

Methods

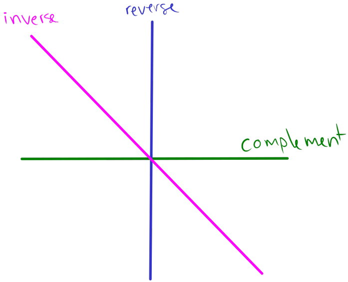
At the time, FindStat included 387 permutation statistics and 19 bijective maps on permutations. Using the interface with SageMath computational software, we tested all combinations of these maps and statistics, finding 117 potential instances of homomesy, involving 9 maps and 68 statistics. We proved the following:

- Lehmer code rotation (45 homomesic statistics)
- Complement and reverse (22 statistics homomesic for both maps, 5 statistics homomesic for reverse but not complement, and 13 statistics homomesic for complement but not reverse)
- Foata bijection and variations (4 maps all having the same single homomesic statistic)
- Kreweras complement and inverse Kreweras complement (3 homomesic statistics)

Reverse, inverse and complement

New composition relationships:

- $\mathcal{R}(\sigma^{-1}) = \mathcal{C}(\sigma)^{-1}$
- $(\mathcal{R} \circ \mathcal{C})^2 = e$
- $(\mathcal{R} \circ \mathcal{I})^4 = e$, where $\mathcal{I}(\sigma) = \sigma^{-1}$



Remark

No statistic in FindStat is homomesic under the inverse map.
The reverse and complement will only ever have one fixed point iff n is odd.

A more interesting map: the Lehmer code rotation

MapsDatabase

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Identifier Mp00149: Permutations —Lehmer code rotation⟶ Permutations

Images

```
[] => []
[1] => [1]
[1,2] => [2,1]
[2,1] => [1,2]
[1,2,3] => [2,3,1]
[1,3,2] => [2,1,3]
[2,1,3] => [3,2,1]
[2,3,1] => [3,1,2]
[3,1,2] => [1,3,2]
[3,2,1] => [1,2,3]
[1,2,3,4] => [2,3,4,1]
[1,2,4,3] => [2,3,1,4]
[1,3,2,4] => [2,4,3,1]
```

click to show experimental identities (only identities of compositions of up to three maps are shown)

Description Sends a permutation π to the unique permutation τ (of the same length) such that every entry in the Lehmer code of τ is cyclically one larger than the Lehmer code of π.

Properties bijective, graded

Sage code

```
def mapping(pi):
    lc = pi.to_lehmer_code()
    n = len(lc)
    lc = [ (c+1) % (n-i) for i,c in enumerate(lc) ]
    return sage.combinat.permutation.from_lehmer_code(lc)
```

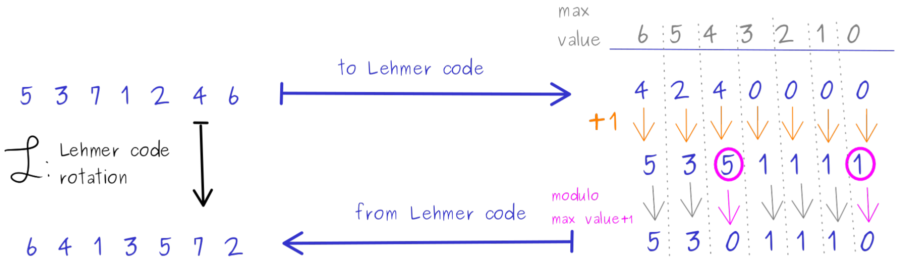
Weight 26

Created Sep 03, 2020 at 13:41 by **Christian Stump**

Updated Sep 03, 2020 at 13:41 by **Christian Stump**

A more interesting map: the Lehmer code rotation

The *Lehmer code* of a permutation encodes the number of inversions starting at a given position.



Sample result: Lehmer code rotation

1234	(0, 0, 0, 0)	2134	(1, 0, 0, 0)
2341	(1, 1, 1, 0)	3241	(2, 1, 1, 0)
3412	(2, 2, 0, 0)	4312	(3, 2, 0, 0)
4132	(3, 0, 1, 0)	1243	(0, 0, 1, 0)
1324	(0, 1, 0, 0)	2314	(1, 1, 0, 0)
2431	(1, 2, 1, 0)	3421	(2, 2, 1, 0)
3124	(2, 0, 0, 0)	4123	(3, 0, 0, 0)
4231	(3, 1, 1, 0)	1342	(0, 1, 1, 0)
1423	(0, 2, 0, 0)	2413	(1, 2, 0, 0)
2143	(1, 0, 1, 0)	3142	(2, 0, 1, 0)
3214	(2, 1, 0, 0)	4213	(3, 1, 0, 0)
4321	(3, 2, 1, 0)	1432	(0, 2, 1, 0)

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All orbits have size $\mathrm{lcm}(1, 2, 3, \ldots, n)$.

Lehmer code rotation results

Theorem (E.-Lafrenière-McNicholas-Striker-Welch)

45 statistics are homomesic under the Lehmer code rotation, including:

- *right-to-left minima, not left-to-right minima*
- *descents*
- *inversions*
- *various permutation patterns*
- 1^{st} *entry, not last*
- *rank*

Rank is homomesic for reverse and Lehmer code rotation

The *rank* of a permutation is its rank among the permutations of n ordered lexicographically.

σ	123	132	213	231	312	321
$L(\sigma)$	000	010	100	110	200	210

Observation

Sorting lexicographically following the Lehmer code is equivalent to sorting lexicographically following the permutations.

Rank is homomesic for reverse and Lehmer code rotation

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Proposition

The rank is given by $\mathrm{rank}(\sigma) = 1 + \sum_{i=1}^{n-1} L(\sigma)_i (n-i)!$*, where* $L(\sigma)_i$ *is the* i*-th entry of the Lehmer code of* σ*.*

Theorem

The rank is homomesic for the Lehmer code rotation and the reverse.

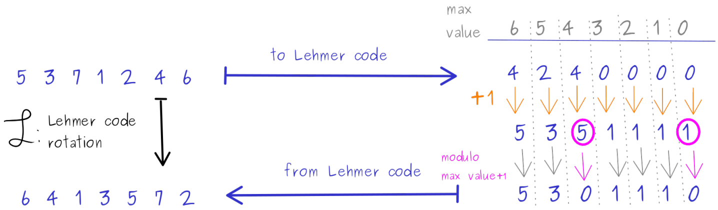
What is Lehmer code rotation?

Remark

We tested several maps dictated by the Lehmer code, and none of them exhibited homomesy!

Question

What is the geometric meaning of the Lehmer code rotation?



Thank you for your time

Flattened Stirling Permutations appeared in Integers.

Homomesies on permutations appeared in Mathematics of Computation, and a preprint is available on the arXiv.