Solving puzzles of shellable simplicial spheres

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Slides

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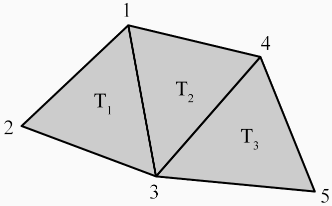
“Shellable”?

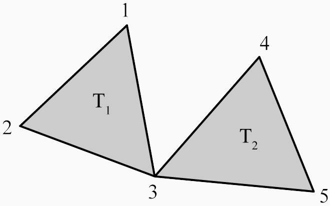
Let Δ be a ***pure*** simplicial complex.

A ***shelling*** of Δ is an ordering $T_1, \dots, T_n$ of the facets of Δ such that $\overline{T_i} \cap (\overline{T_1} \cup \cdots \cup \overline{T_{i-1}})$ is a pure $(\dim \Delta - 1)$-dimensional simplicial complex for every $2 \leq i \leq n$.

Δ is ***shellable*** if such a shelling exists.



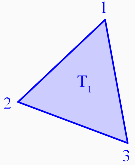
T_1, T_2, T_3 is a shelling.



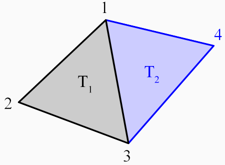
T_1, T_2 is not a shelling.
$\dim \overline{T_1} \cap \overline{T_2} = \dim \overline{3} = 0 < 1$.

“Shellable”?

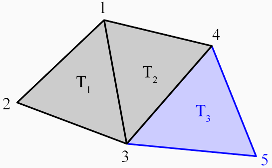
Every shelling gives a ***partitioning*** of the complex.



Minimal new face of T_1: $\varnothing$.



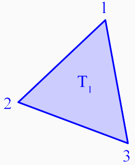
Minimal new face of T_2: 4.



Minimal new face of T_3: 5.

“Shellable”?

Every shelling gives a ***partitioning*** of the complex.



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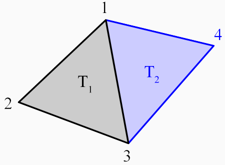
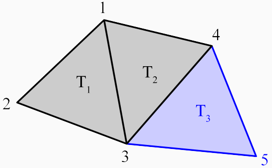
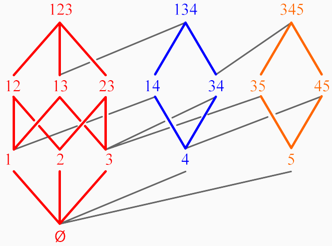


Minimal new face of T_2: 4.



Minimal new face of T_3: 5.

$\Longrightarrow$ Partitioning:
$[\varnothing, 123] \sqcup [4, 134] \sqcup [5, 345]$.



“Simplicial sphere”?

Δ is a ***simplicial*** $(d-1)$***-sphere*** if its ***geometric realization*** is homeomorphic to a topological $(d-1)$-sphere.

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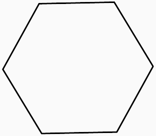
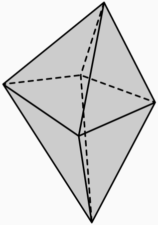
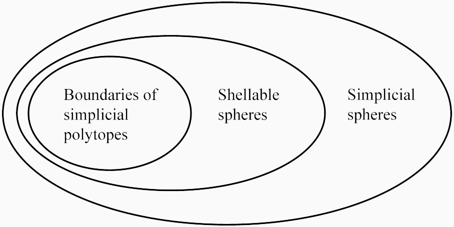




Simplicial 1-spheres = cycles.
Simplicial 2-spheres = boundaries of simplicial 3-polytopes (Steinitz’s theorem).
They are all shellable.

“Simplicial sphere”?



Theorem (Goodman, Pollack, 1986 [4]; Alon, 1986 [1])

There are $2^{\Theta(n \log n)}$ combinatorially distinct d-polytopes with n vertices for $d \geq 4$.

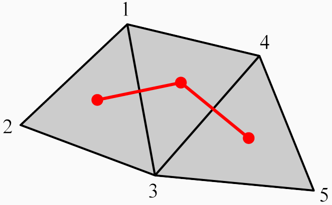
Theorem (Kalai, 1988 [8]; Lee, 2000 [10]; Nevo, Santos, Wilson, 2016 [11]; Benedetti, Ziegler, 2011 [2]; Stanley, 1975 [12]; Y., 2024 [13])

There are $2^{\Theta(n^{\lceil (d-1)/2 \rceil})}$ combinatorially distinct shellable $(d-1)$-spheres with n vertices for $d \geq 4$.

“Puzzle”?

The ***facet-ridge graph (puzzle)*** of Δ is a graph G where

- vertices represent the facets of Δ,
- two vertices form an edge $\iff$ the corresponding facets share a ridge.



"Puzzle"?

Conjecture (Kalai, 2009 [7])

Every **simplicial sphere** is completely determined by its facet-ridge graph.

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Theorem (Blind and Mani-Levitska, 1987 [3]; Kalai, 1988 [9])

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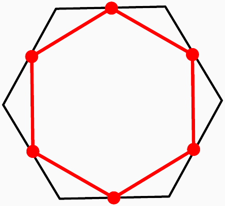
The facet-ridge graph of a 1-sphere is isomorphic to the sphere itself.



“Puzzle”?

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We know that G is the facet-ridge graph of some **shellable** $(d-1)$-sphere Δ. The goal is to recover the combinatorial structure of Δ.

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Language: k-frames and k-systems (Joswig, Kaibel, Körner, 2002 [6]).

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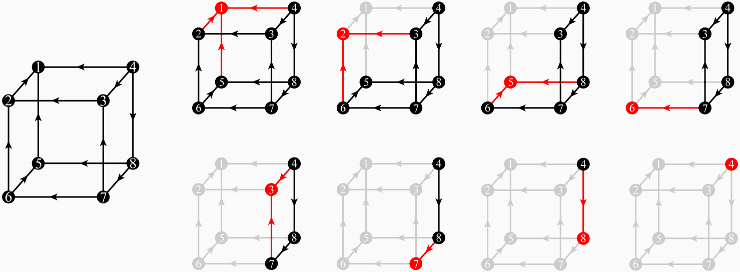
Repeatedly taking sinks of the graph to get a shelling of Δ.



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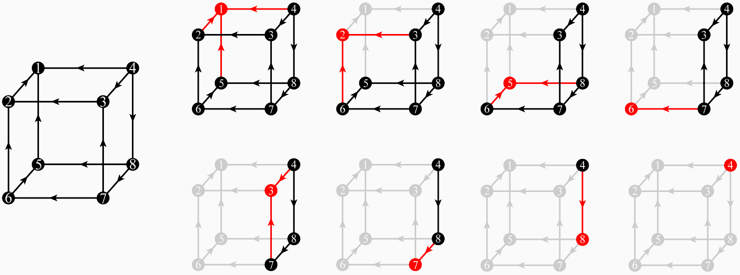
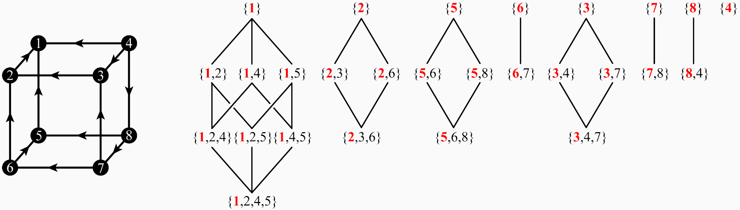


The corresponding partitioning:



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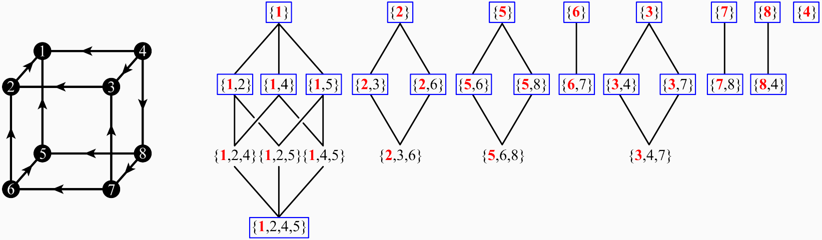

Observation

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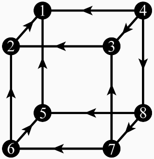

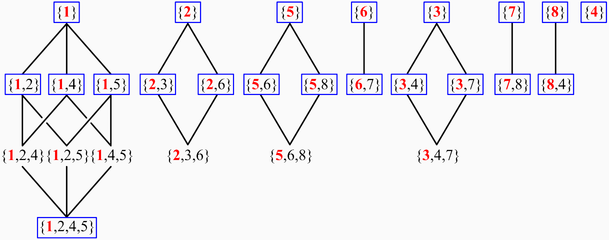

Observation

For $(d-1)$-faces (facets): contained only in itself.

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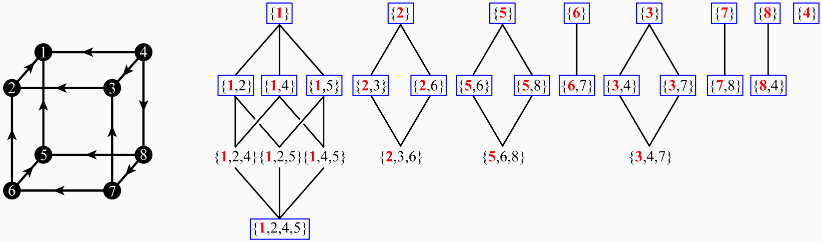

Observation

For $(d-1)$-faces (facets): contained only in itself.

For $(d-2)$-faces (ridges): contained in exactly two facets, indicated by G.

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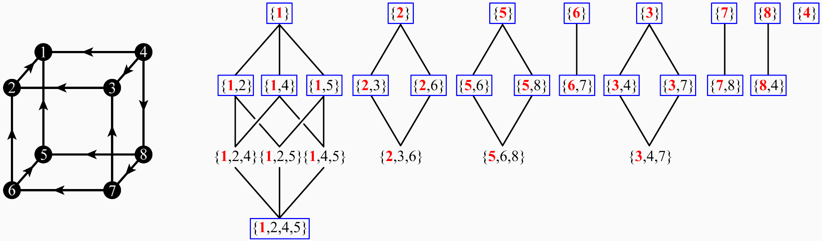

Observation

For $(d-1)$-faces (facets): contained only in itself.

For $(d-2)$-faces (ridges): contained in exactly two facets, indicated by G.

For the (-1)-face ($\varnothing$): contained in every facet.

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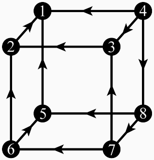

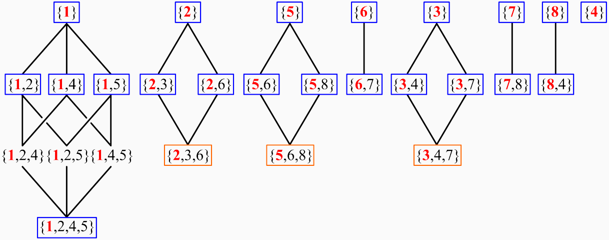

Lemma

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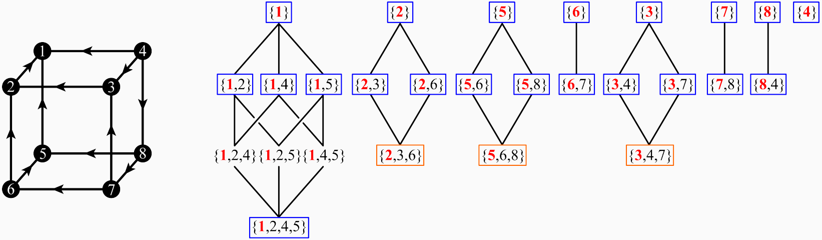

Lemma

- For every **minimal new face** in the shelling, we can determine which facets contain it.

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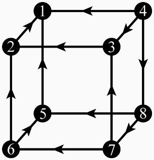

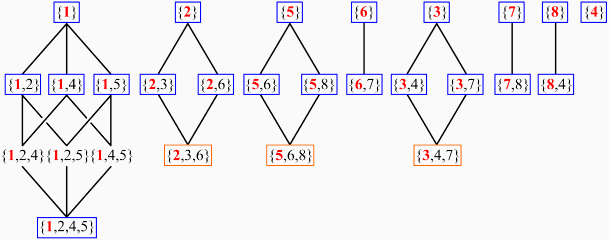

Lemma

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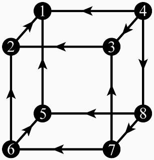

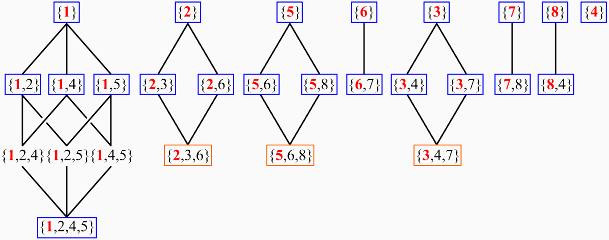

Lemma

- For every **minimal new face** in the shelling, we can determine which facets contain it.
- If for each face NOT in T_1, we know which facets contain it, then we can also determine such information for every face in T_1.

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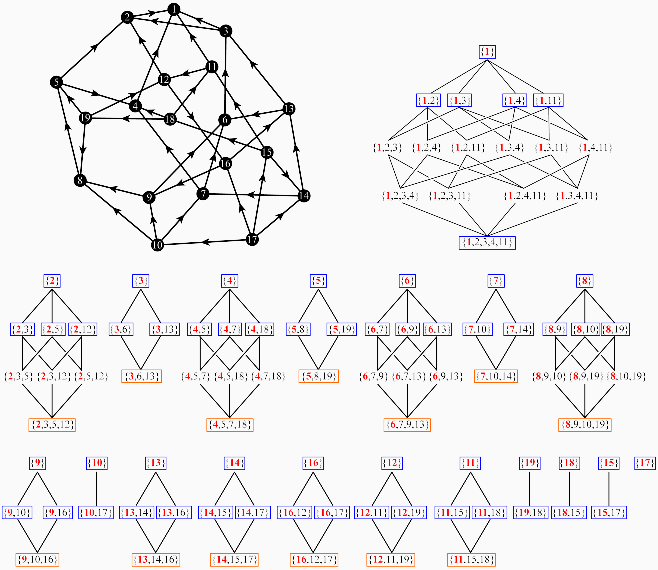

A 3-dimensional example.

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- Partitionable spheres? (Is there a non-partitionable simplicial sphere?)

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- Flag spheres?

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- Does the facet-ridge graph determine the f-vector of a nonshellable sphere? (Or more generally, of any Cohen–Macaulay manifold?)

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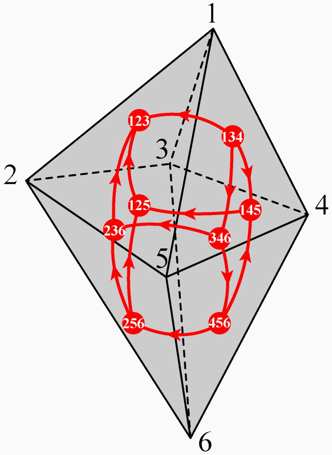

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Let $M = \min\{f^{\mathcal{O}} : \mathcal{O}$ is an acyclic orientation of $G\}$. Then M is the total number of faces of Δ, and $\mathcal{O}$ is good if and only if $f^{\mathcal{O}} = M$.

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$f^{\mathcal{O}} = h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + \cdots + 2^d h_d^{\mathcal{O}}$.

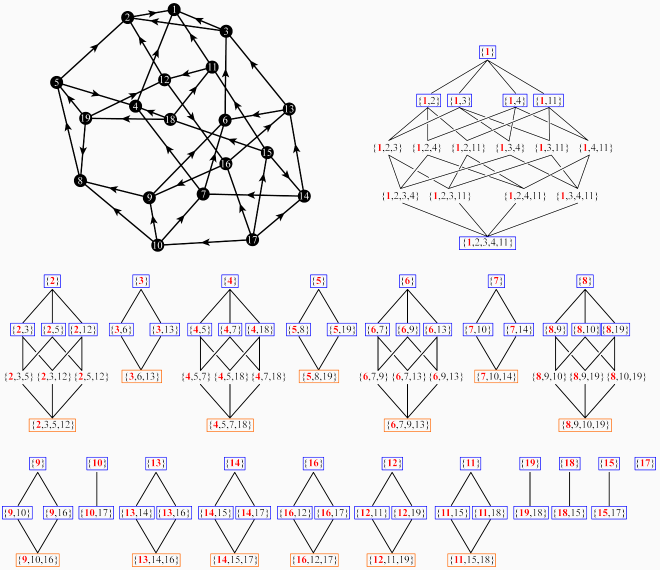
Proposition (Essentially due to Kalai [9])

Let $M = \min\{f^{\mathcal{O}} : \mathcal{O} \text{ is an acyclic orientation of } G\}$. Then M is the total number of faces of Δ, and $\mathcal{O}$ is good if and only if $f^{\mathcal{O}} = M$.

Key idea in the proof.

$f^{\mathcal{O}} =$ # (star, sink) pairs $\geq$ # stars = # faces.

Step 3. For every face σ of Δ, find all facets that contain σ.



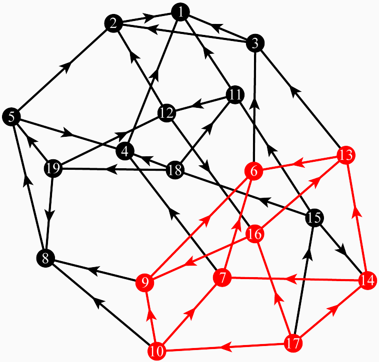
A 3-dimensional example.

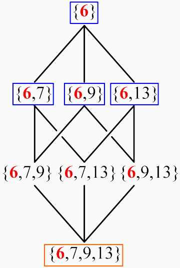
Step 3. For every face σ of Δ, find all facets that contain σ.

Fact

Define
$\mathcal{V}_\Delta(\sigma) := \{t \in V(G) : \text{the corresponding facet } T \text{ of } t \text{ contains } \sigma\}$.

For a non-empty face $\sigma \in \Delta$, $G[\mathcal{V}_\Delta(\sigma)]$ is the facet-ridge graph of a lower dimensional shellable sphere $\mathrm{lk}_\Delta \sigma$.





$$\mathcal{V}_\Delta(\{6, 7, 9, 13\}) = \{6, 7, 9, 10, 13, 14, 16, 17\}.$$