Oriented Matroids from Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$

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**Oriented Matroid**: An abstraction of linear (in)dependence over $\mathbb{R}$.

**Intuition**: Given a $d \times n$ real matrix $A$. Then $\forall |X| = d - 1, |Y| = d + 1,$

$$
\sum_{k=1}^{d+1} (-1)^k \det(A|_{X,y_k}) \det(A|_{Y\setminus y_k}) = 0.
$$

Let $E$ be the column set and $\chi(i_1, \ldots, i_d) = \text{sign} \det(A_{i_1} \ldots A_{i_d})$.

**Definition**

A *chirotope* is a (non-zero) map $\chi : E^d \to \{+, -, 0\}$ that is

- alternating;
- Grassmann–Plücker: $(-1)^k \chi(X, y_k) \chi(Y \setminus y_k)$’s either contain both a +ve and a -ve term, or are all zeros.
Topological Representation Theorem

Each column $A_i$ defines a hyperplane $A_i^\perp \subset \mathbb{R}^d$.

**Theorem (Folkman–Lawrence 1978)**

Oriented Matroids $\iff$ Pseudosphere Arrangements.
Convex Geometry: real hyperplane arrangements, polytopes

Algebraic Geometry: strata of real Grassmannians (Mnëv’s universality theorem)

Topology: real vector bundles and their characteristic classes

Optimization: linear programming (simplex method) and beyond
Matching Fields

What if instead of \( \det(A|_{\sigma}) \)'s, we only compute one term per \( \det(A|_{\sigma}) \)?

**Notation:** Entries of \( A \leftrightarrow \) Edges of \( K_{R,E} \), with \( |R| = d, |E| = n \).

**Definition**

*Matching Field:* A collection of perfect matchings, one \( M_\sigma \) between \( R \) and \( \sigma \) for every \( \sigma \subset E \) of size \( d \).

Given a *nowhere zero* sign matrix \( A \), set \( \chi(\sigma) := \text{sign}(M_\sigma) \prod_{e \in M_\sigma} A_e \).

**Example:** Take the max. perfect matchings w.r.t. generic weights.

\[
\begin{pmatrix}
+1.8 & -0.6 & -0.9 \\
-1.2 & -1.6 & +2.2 \\
+2 & -1.4 & +0.2
\end{pmatrix}, \quad \chi = (-1)(1 \cdot -1 \cdot 1) = \text{sign} \det \begin{pmatrix}
+e^{18} & -e^6 & -e^9 \\
-e^{12} & -e^{16} & +e^{22} \\
+e^{20} & -e^{14} & +e^2
\end{pmatrix}
\]

**Motivation:** Tropical geometry & Gröbner theory [Sturmfels–Zelevinsky 93].
Triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

**Notation:** Vertices of $\Delta_{d-1} \times \Delta_{n-1} \leftrightarrow$ Edges of $K_{R,E}$.

**Proposition**

The vertices of any full-dim simplex in $\Delta_{d-1} \times \Delta_{n-1}$ form a spanning tree.
Fix a triangulation. Take all perfect matchings that are subgraphs of some trees. This gives a polyhedral matching field.

Observation

*If the triangulation is regular, then we get back the tropical example.*
Why Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$?

**Reason I:** They appear in many places!

- **Algebraic Geometry:** toric Hilbert schemes, Schubert calculus
- **Tropical Geometry:** tropical convexity, Stiefel tropical linear spaces
- **Optimization:** tropical linear programming, mean payoff game
- **Tropical pseudohyperplane arrangements, tropical oriented matroids, trianguloids, etc**

**Reason II:** Correct direction in view of [Sturmfels–Zelevinsky].

Coherent $\subsetneq$ Polyhedral $\subsetneq$ Linkage
Main Theorem

Theorem (Celaya–Loho–Y. 2020+)

*Polyhedral matching fields induce uniform oriented matroids.*

Proof Sketch: Divide

**Divide:** The triangulation induces a *matroid subdivision* of the hypersimplex by *transversal matroid polytopes* (of the trees).

**Definition**

*Matroid polytope:* \( \text{conv}\{e_B : B \in \mathcal{B}(M)\} \).

*Matroid subdivision:* Subdivision of a MP by MPs.

*Transversal matroid:* \( \sigma \subset E \) is a basis iff \( \exists R \equiv \sigma \) perfect matching in \( T \).
Proof Sketch: Conquer and Merge

**Conquer:** Each restriction is a chirotope (realizable by $A$ restricted to the edges of the tree).

**Merge:**

**Lemma (Celaya–Loho–Y.)**

Let $\chi : \mathcal{B}(M) \to \{+, -\}$ and $M_1, \ldots, M_k$ be a matroid subdivision of $M$. If every $\chi_{M_i}$ is a chirotope, then $\chi$ is also a chirotope.

**Proof:** Reduce to 3-term GP and analyze the subdivision on 3-dim faces.

**Definition**

3-term GP relation: $\forall a, b, c, d \in E,$

$\chi(a, b, _)\chi(c, d, _), -\chi(a, c, _)\chi(b, d, _), \chi(a, d, _)\chi(b, c, _),$ either contain both a $+$ve and a $-$ve term, or are all zeros.
Viro’s Patchworking

Given a *regular* triangulation of $n\triangle_{d-1}$ and signs assigned at the vertices. Take the “zero locus” within each cell, and glue all loci together.

**Theorem (Viro 1980’s)**

*The locus is isotopic to some real algebraic hypersurface.*
Using *Cayley trick*, convert a triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ into a *fine mixed subdivisions* of $n\triangle_{d-1}$.

**Theorem (Celaya–Loho–Y. 2020+)**

The locus is a *pseudosphere arrangement* representing $\chi$. 
Some Proof Ingredients

Combinatorics: The face poset is the \textit{covector lattice}.

- Faces $\rightarrow$ Signed Forests $\rightarrow$ Covectors: Restrict to individual cells.
- Surjectivity: Borsuk–Ulam + Topological Representation Theorem.

Topology: The CW complex is \textit{regular}.

- View patchworking as a stepwise cell merging.
- Show that every step preserves regularity.
Future Directions

Question

*Can all OMs be realized by triangulations? If not, which?*

Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ are to tropical LP as oriented matroids are to linear programming. Implications in optimization algorithms and complexity theory?

Question

*What else can we do with signed triangulations and matroid subdivisions?*

Question

*Do we always get strong matroids for $\mathbb{H}$ with the inflation property?*
Thank you!


_______________. *Patchworking Oriented Matroids*. To be splitted from the above.
A *hyperfield* is “a field with a multi-valued addition”.

**Example** (Sign hyperfield): $\mathbb{S} = \{+, -, 0\}$, $+ \boxplus - = \{+, -, 0\}$.

**Definition (Baker–Bowler 2017)**

A **strong** matroid over $\mathbb{H}$ is an alternating $\chi : E^d \to \mathbb{H}$ such that

$$0 \in \boxplus_{k=1}^{d+1} (-1)^k \chi(X, y_k) \otimes \chi(Y \setminus y_k).$$

A **weak** matroid only requires the 3-term GP as long as $\chi$ is a matroid.

**Example**: Oriented matroids = Matroids over $\mathbb{S}$.

Also linear subspaces, matroids, valuated matroids, phase matroids...

**Caution**: In general, $\{\text{Strong matroids}\} \subsetneq \{\text{Weak matroids}\}$. 
Definition (Anderson–Eppolito; Massouros)

*Inflation property:* \(1 \boxplus (-1) = \mathbb{H} \).

Theorem (Celaya–Looho–Y. 2020+)

*Suppose \(\mathbb{H} \) has the IP. Then given a polyhedral \(\{M_\sigma\}\) and a nowhere zero \(\mathbb{H}\)-matrix \(A\), \(\chi(\sigma) := \text{sign}(M_\sigma) \bigotimes_{e \in M_\sigma} A_e\) is a weak matroid over \(\mathbb{H}\).*

- This characterizes hyperfields that have the IP, but the theorem is true for *any* \(\mathbb{H}\) up to “perturbation”. 