Oriented Matroids from Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$

Chi Ho Yuen

Brown University

Joint Work with Marcel Celaya (TU Berlin) and Georg Loho (LSE)

AMS Fall Central Sectional Meeting

September 12, 2020

A Crash Course in Oriented Matroids

Oriented Matroid: An abstraction of linear (in)dependence over $\mathbb{R}$.

Intuition: Given a $d \times n$ real matrix A. Then $\forall\ |X| = d-1, |Y| = d+1$,

$$\sum_{k=1}^{d+1} (-1)^k \det(A|_{X,y_k}) \det(A|_{Y \setminus y_k}) = 0.$$

Let E be the column set and $\chi(i_1, \ldots, i_d) = \operatorname{sign} \det(A_{i_1} \ldots A_{i_d})$.

Definition

A *chirotope* is a (non-zero) map $\chi : E^d \to \{+, -, 0\}$ that is

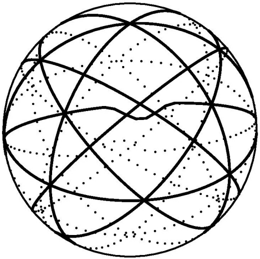
- alternating;
- Grassmann–Plücker: $(-1)^k \chi(X, y_k)\chi(Y \setminus y_k)$'s either contain both a $+$ve and a -ve term, or are all zeros.

Topological Representation Theorem

Each column A_i defines a hyperplane $A_i^\perp \subset \mathbb{R}^d$.

Theorem (Folkman–Lawrence 1978)

Oriented Matroids $\Leftrightarrow$ Pseudosphere Arrangements.



Oriented Matroids in Mathematics

- Convex Geometry: real hyperplane arrangements, polytopes
- Algebraic Geometry: strata of real Grassmannians (Mnëv's universality theorem)
- Topology: real vector bundles and their characteristic classes
- Optimization: linear programming (simplex method) and beyond

Matching Fields

What if instead of $\det(A|_\sigma)$'s, we only compute one term per $\det(A|_\sigma)$?

Notation: Entries of $A \Leftrightarrow$ Edges of $K_{R,E}$, with $|R| = d, |E| = n$.

Definition

Matching Field: A collection of perfect matchings, one M_σ between R and σ for every $\sigma \subset E$ of size d.
Given a *nowhere zero* sign matrix A, set $\chi(\sigma) := \mathrm{sign}(M_\sigma) \prod_{e \in M_\sigma} A_e$.

EXAMPLE: Take the max. perfect matchings w.r.t. generic weights.

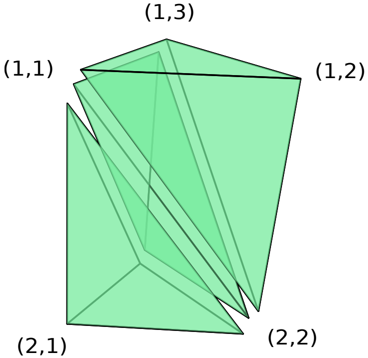
$$\begin{pmatrix} +\mathbf{1.8} & -0.6 & -0.9 \\ -1.2 & -1.6 & +\mathbf{2.2} \\ +2 & -\mathbf{1.4} & +0.2 \end{pmatrix}, \chi = (-1)(1 \cdot -1 \cdot 1) = \mathrm{sign} \det \begin{pmatrix} +\mathbf{e^{18}} & -e^{6} & -e^{9} \\ -e^{12} & -e^{16} & +\mathbf{e^{22}} \\ +e^{20} & -\mathbf{e^{14}} & +e^{2} \end{pmatrix}$$

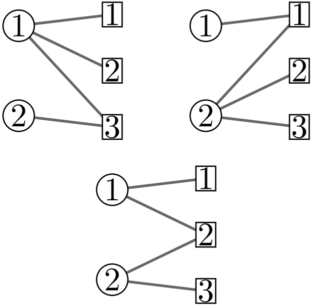
Motivation: Tropical geometry & Gröbner theory [Sturmfels–Zelevinsky 93].

Notation: Vertices of $\triangle_{d-1} \times \triangle_{n-1} \Leftrightarrow$ Edges of $K_{R,E}$.

Proposition

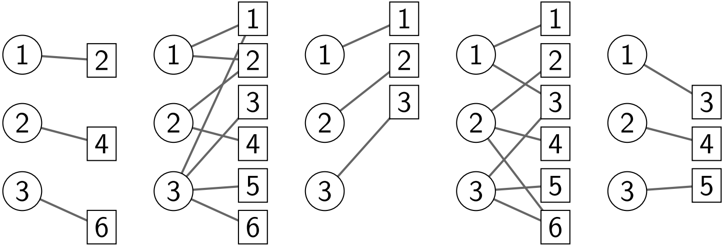
The vertices of any full-dim simplex in $\triangle_{d-1} \times \triangle_{n-1}$ form a spanning tree.





Polyhedral Matching Fields

Fix a triangulation. Take all perfect matchings that are subgraphs of some trees. This gives a *polyhedral matching field*.



Observation

If the triangulation is regular, then we get back the tropical example.

Why Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$?

Reason I: They appear in many places!

- Algebraic Geometry: toric Hilbert schemes, Schubert calculus
- Tropical Geometry: tropical convexity, Stiefel tropical linear spaces
- Optimization: tropical linear programming, mean payoff game
- Tropical pseudohyperplane arrangements, tropical oriented matroids, trianguloids, etc

Reason II: Correct direction in view of [Sturmfels–Zelevinsky].

$$\text{Coherent} \subsetneq \text{Polyhedral} \subsetneq \text{Linkage}$$

Main Theorem

Theorem (Celaya–Loho–Y. 2020+)

Polyhedral matching fields induce uniform oriented matroids.

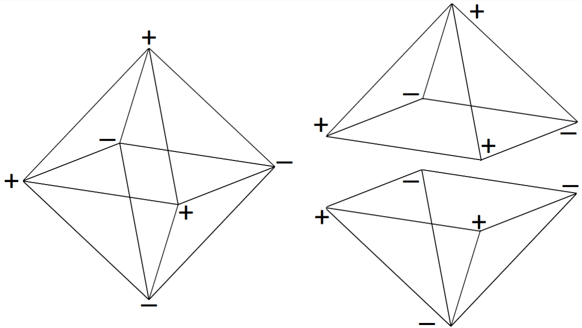
Proof Strategy: Divide-and-Conquer.

Proof Sketch: Divide

Divide: The triangulation induces a *matroid subdivision* of the hypersimplex by *transversal matroid polytopes* (of the trees).

Definition

Matroid polytope: $\text{conv}\{\mathbf{e}_B : B \in \mathcal{B}(M)\}$.
Matroid subdivision: Subdivision of a MP by MPs.
Transversal matroid: $\sigma \subset E$ is a basis iff $\exists$ $R \equiv \sigma$ perfect matching in T.



Proof Sketch: Conquer and Merge

Conquer: Each restriction is a chirotope (realizable by A restricted to the edges of the tree).

Merge:

Lemma (Celaya–Loho–Y.)

Let $\chi : \mathcal{B}(M) \to \{+,-\}$ and $M_1, \ldots, M_k$ be a matroid subdivision of M. If every χ_{M_i} is a chirotope, then χ is also a chirotope.

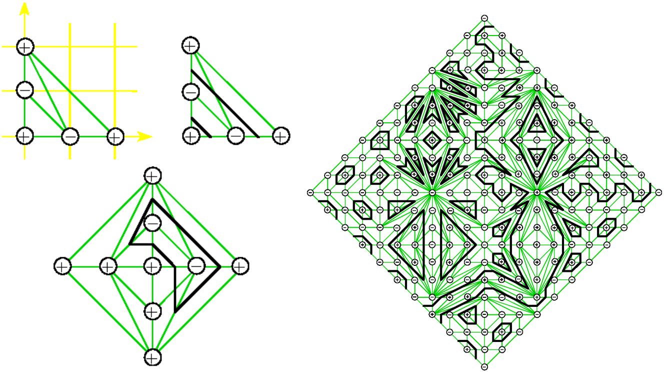
PROOF: Reduce to 3-term GP and analyze the subdivision on 3-dim faces.

Definition

3-term GP relation: $\forall a, b, c, d \in E$,
$\chi(a, b, \_)\chi(c, d, \_), -\chi(a, c, \_)\chi(b, d, \_), \chi(a, d, \_)\chi(b, c, \_)$,
either contain both a $+$ve and a $-$ve term, or are all zeros.

Viro's Patchworking

Given a *regular* triangulation of $n\triangle_{d-1}$ and signs assigned at the vertices. Take the "zero locus" within each cell, and glue all loci together.

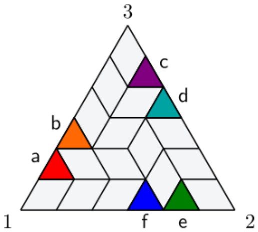


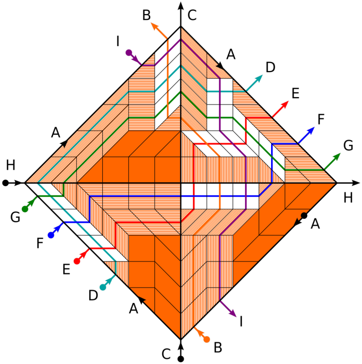
Theorem (Viro 1980's)

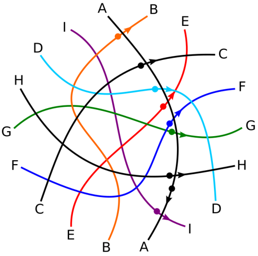
The locus is isotopic to some real algebraic hypersurface.

Patchworking Oriented Matroids

Using *Cayley trick*, convert a triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ into a *fine mixed subdivisions* of $n\triangle_{d-1}$.







Theorem (Celaya–Loho–Y. 2020+)

The locus is a pseudosphere arrangement representing χ.

Some Proof Ingredients

Combinatorics: The face poset is the *covector lattice*.

- Faces $\Rightarrow$ Signed Forests $\Rightarrow$ Covectors: Restrict to individual cells.
- Surjectivity: Borsuk–Ulam + Topological Representation Theorem.

Topology: The CW complex is *regular*.

- View patchworking as a stepwise cell merging.
- Show that every step preserves regularity.

Future Directions

Question

Can all OMs be realized by triangulations? If not, which?

Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ are to tropical LP as oriented matroids are to linear programming.
Implications in optimization algorithms and complexity theory?

Question

What else can we do with signed triangulations and matroid subdivisions?

Question

Do we always get strong matroids for $\mathbb{H}$ with the inflation property?

Thank you!

Marcel Celaya, Georg Loho, Chi Ho Yuen. *Oriented Matroids from Triangulations of Products of Simplices.* arXiv:2005.01787.

________________. *Patchworking Oriented Matroids.* To be splitted from the above.

Matroids over Hyperfields

A *hyperfield* is "a field with a multi-valued addition".

EXAMPLE (Sign hyperfield): $\mathbb{S} = \{+, -, 0\}$, $+ \boxplus - = \{+, -, 0\}$.

Definition (Baker–Bowler 2017)

A **strong** matroid over $\mathbb{H}$ is an alternating $\chi : E^d \to \mathbb{H}$ such that

$$0 \in \boxplus_{k=1}^{d+1} (-1)^k \chi(X, y_k) \otimes \chi(Y \setminus y_k).$$

A **weak** matroid only requires the 3-term GP as long as $\underline{\chi}$ is a matroid.

EXAMPLE: Oriented matroids = Matroids over $\mathbb{S}$.

Also linear subspaces, matroids, valuated matroids, phase matroids...

Caution: In general, $\{\text{Strong matroids}\} \subsetneq \{\text{Weak matroids}\}$.

Our Theorem for Matroids over Hyperfields

Definition (Anderson–Eppolito; Massouros)

Inflation property: $1 \boxplus (-1) = \mathbb{H}$.

Theorem (Celaya–Loho–Y. 2020+)

Suppose $\mathbb{H}$ has the IP. Then given a polyhedral $\{M_\sigma\}$ and a nowhere zero $\mathbb{H}$-matrix A, $\chi(\sigma) := \mathrm{sign}(M_\sigma) \bigotimes_{e \in M_\sigma} A_e$ is a **weak** *matroid over $\mathbb{H}$.*

- This characterizes hyperfields that have the IP, but the theorem is true for *any* $\mathbb{H}$ up to "perturbation".