Polytopal Realizations and Faces of Extended Nestohedra

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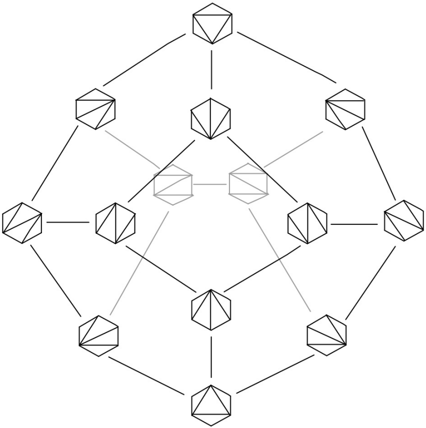
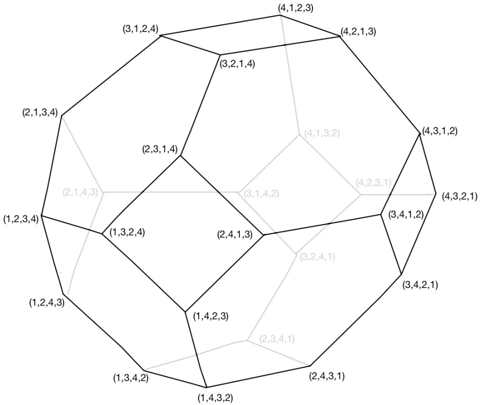
Associahedron and Permutohedron





Vastly generalize to **nestohedra**

Nestohedra Results

Simple convex polytopes	Multiple realizations, e.g., Postnikov '09
Gal's conjecture holds flag nestohedra	Volodin '10
Combinatorial interpretation for γ-vector for subclass of flag nestohedra	Postnikov–Reiner–Williams '08
Volumes and Ehrhart polynomials	Postnikov '09
Some are dual to cluster complexes for cluster algebras	Fomin–Zelevinsky '03

Motivation for Extended Nestohedron

- Lam–Pylyavskyy introduced **Laurent phenomenon (LP) algebras** (2012), generalizing cluster algebras
- Examined cluster complex of linear LP algebras and conjectured that the complex is a simplicial polytope
- **Goal**: study this complex and its dual (extended nestohedron), and compare with the nestohedron

Building Sets

Definition

A **(connected) building set** $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that

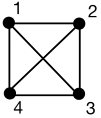
1. $\mathcal{B}$ contains all singletons $\{i\}$ for all $i \in [n]$, and $[n]$
2. if $I, J \in \mathcal{B}$ with $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.

If Γ is a graph on vertex set $[n]$, then the associated **graphical building set** $\mathcal{B}_\Gamma$ is defined to be

$$\mathcal{B}_\Gamma = \{I \subseteq [n] : \Gamma|_I \text{ is connected}\}.$$

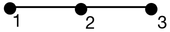
Building Set Examples

Complete graph K_n



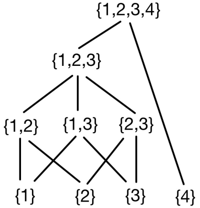
- all subsets of $[n]$
- $\mathcal{B}_{K_4} = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 234, 124, 134, 1234\}$

Path graph P_n



- all interval subsets of $[n]$
- $\mathcal{B}_{P_3} = \{1, 2, 3, 12, 23, 123\}$

Non-graphical building set



- $\mathcal{B} = \{1, 2, 3, 4, 12, 13, 23, 123, 1234\}$

Nested Collections

Definition

For a building set $\mathcal{B}$, a **nested collection** N of $\mathcal{B}$ is a collection of elements $\{I_1, \ldots, I_m\} \subseteq \mathcal{B} \setminus [n]$ such that

1. for any $i \neq j$, I_i and I_j are either nested or disjoint, and
2. for any $I_{i_1}, \ldots, I_{i_k}$ pairwise disjoint, their union is not an element of $\mathcal{B}$.

Example

Consider $\mathcal{B} = \mathcal{B}_{P_4} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$.

- $\{1, 3, 34\}$ is a nested collection.
- $\{1, 2, 23\}$ **is not** a nested collection since $\{1\} \cup \{2\} \in \mathcal{B}$.

Nested Complexes and Nestohedra

Definition

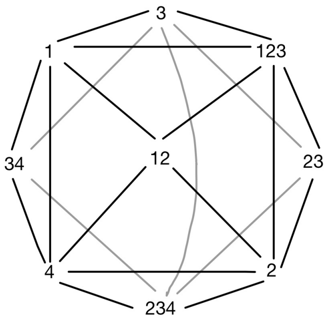
For a building set $\mathcal{B}$ on $[n]$, the **nested complex** $\mathcal{N}(\mathcal{B})$ is the simplicial complex with

- vertices $\{I \mid I \in \mathcal{B} \setminus [n]\}$
- faces $\{I_1, \ldots, I_m\}$ that are nested collections of $\mathcal{B}$

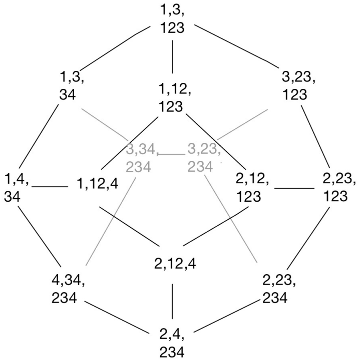
The **nestohedron** $\mathcal{P}(\mathcal{B})$ is the polytope polar dual to the nested complex $\mathcal{N}(\mathcal{B})$.

Associahedron Revisited

$\mathcal{N}(\mathcal{B}_{P_4})$



$\mathcal{P}(\mathcal{B}_{P_4})$



- $\mathcal{P}(\mathcal{B}_{P_n})$ is the associahedron of order $n+1$
- $\mathcal{P}(\mathcal{B}_{K_n})$ is the permutohedron of order n

Extended Nested Collections

Definition

For a building set $\mathcal{B}$ on $[n]$, an **extended nested collection** $N^\square$ of $\mathcal{B}$ is a collection of elements $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ such that

1. $\{I_1, \ldots, I_m\}$ forms a nested collection of $\mathcal{B}$
   - Can now include $[n]$
2. $i_j \in [n]$ for all j, and $i_j \notin I_k$ for all $1 \leq k \leq m$

Example

$\mathcal{B} = \mathcal{B}_{P_4}$

- $\{1, 3, 34, x_2\}$ is an extended nested collection.
- $\{1, 3, 34, x_4\}$ **is not** an extended nested collection.

Extended Nested Complexes and Nestohedra

Definition

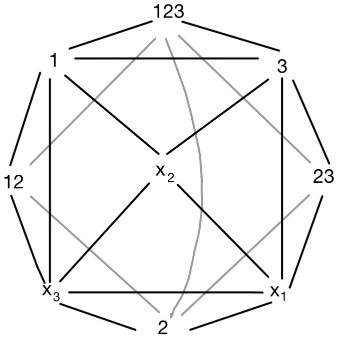
For a building set $\mathcal{B}$ on $[n]$, the **extended nested complex** $\mathcal{N}^\square(\mathcal{B})$ is the simplicial complex with

- vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in [n]\}$
- faces $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ that are extended nested collections of $\mathcal{B}$

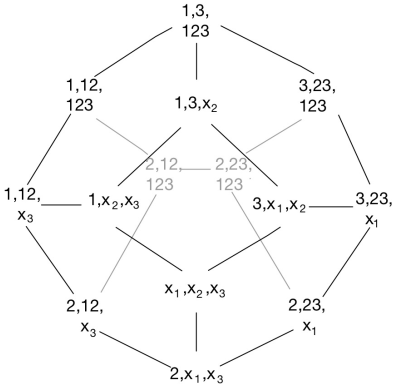
The **extended nestohedron** $\mathcal{P}^\square(\mathcal{B})$ is the polytope dual to the extended nested complex

Associahedron Revisited[2]

$\mathcal{N}^\square(\mathcal{B}_{P_3})$



$\mathcal{P}^\square(\mathcal{B}_{P_3})$



$$\mathcal{P}^\square(\mathcal{B}_{P_{n-1}}) \cong \mathcal{P}(\mathcal{B}_{P_n}) \cong \mathsf{Assoc}_{n+1}$$

What is known so far

Extended nestohedra for graphical $\mathcal{B}$ previously studied

- Devadoss–Heath–Vipismakul introduced **graph cubeahedra** in 2011 (before LP algebras introduced!)
- Manneville–Pilaud '17 studied its dual, called it the **design nested complex**

	Non-extended	Extended ($\square$)
Polytopality	Yes	Yes for graphical $\mathcal{B}$ [DHV11] General $\mathcal{B}$: ?
Gal's conjecture	Yes	?
When is $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$	Known for graphical $\mathcal{B}$ [MP17] General $\mathcal{B}$: ?	

Note: $\{$Extended nestohedron$\} \not\subseteq \{$Nestohedron$\}$

Polytopality

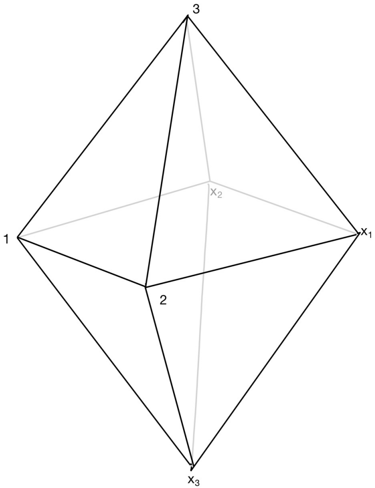
	Non-extended	Extended ($\square$)
Polytopality	Yes	Yes for graphical $\mathcal{B}$ [DHV11] Yes for all $\mathcal{B}$
Gal's conjecture	Yes	?
When is $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$	Known for graphical $\mathcal{B}$ [MP17] General $\mathcal{B}$: ?	

Theorem (REU)

For any building set $\mathcal{B}$, the extended nested complex $\mathcal{N}^{\square}(\mathcal{B})$ can be realized as the boundary of a simplicial convex polytope.

Polytopality

- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices e_i labeled $i \in [n]$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.



$$\mathcal{B}_{P_3} = \{1, 2, 3, 12, 23, 123\}$$

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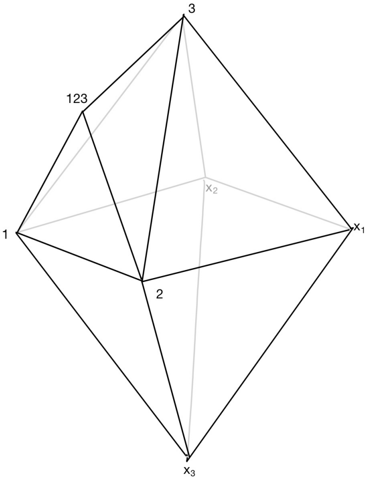
- Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled I.



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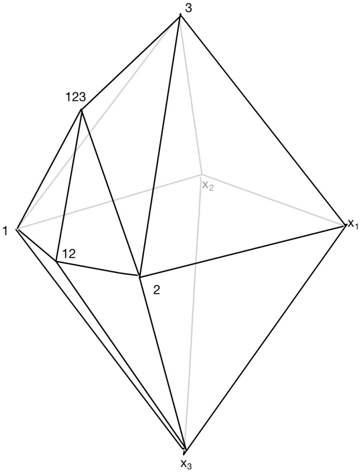

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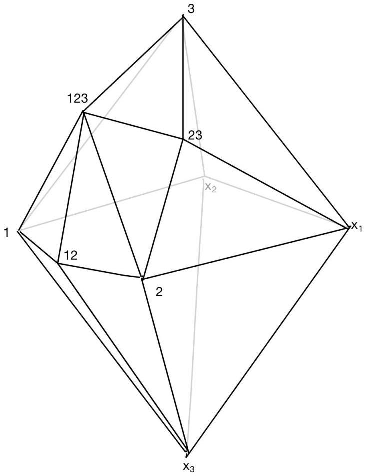

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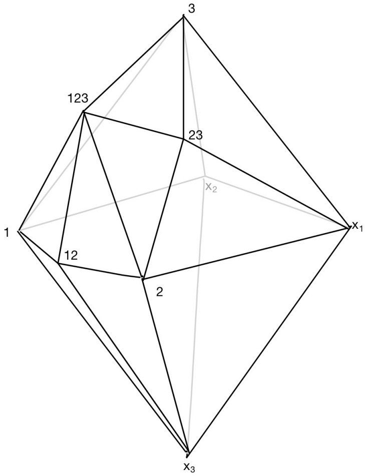
Polytopality

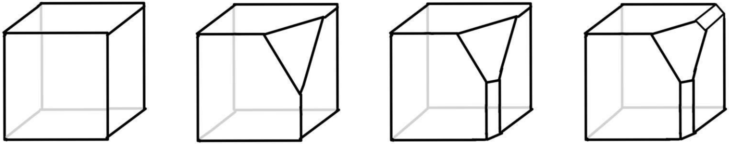
- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \dots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices e_i labeled $i \in [n]$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.
- Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled I.
- The boundary of the resulting polytope is isomorphic to $\mathcal{N}^{\square}(\mathcal{B})$.



$$\mathcal{B}_{P_3} = \{1, 2, 3, 12, 23, 123\}$$

Polytopal Realization

Dually: to realize $\mathcal{P}^{\square}(\mathcal{B})$, start with n-cube and shave faces corresponding to non-singleton building set elements



Gal's Conjecture

Flag polytope: dual to a simplicial complex Δ s.t. collection C of vertices forms simplex in $\Delta \Leftrightarrow$ there's an edge between any two vertices of C in 1-skeleton of Δ

f**-vector**: $f_i =$ number of i-dimensional faces

γ-**vector**: concise encoding of f-vector with smaller integers

Gal's Conjecture

The γ-vector of any flag simple polytope is nonnegative.

Gal's Conjecture for $\mathcal{P}^\square(\mathcal{B})$

	Non-extended	Extended ($\square$)
Polytopality	Yes	Yes for all $\mathcal{B}$
Gal's conjecture	Yes	Yes
When is $\mathcal{N}^\square(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$	Known for graphical $\mathcal{B}$ [MP17] General $\mathcal{B}$: ?	

Gal's Conjecture

The γ-vector of any flag simple polytope is nonnegative.

Gal's Conjecture for $\mathcal{P}^{\Box}(\mathcal{B})$

Theorem (REU)

Gal's conjecture holds for any flag extended nestohedron, i.e., the γ-vector of $\mathcal{P}^{\Box}(\mathcal{B})$ is nonnegative if the polytope is flag.

Proof idea:

- Start with building set $\mathcal{B}$ such that $\mathcal{P}^{\Box}(\mathcal{B})$ is flag
- There exists building set $\mathcal{B}' \subseteq \mathcal{B}$, and $\mathcal{P} = \mathcal{P}^{\Box}(\mathcal{B}')$ has nonnegative γ-vector
- Add back in elements $\mathcal{B} \setminus \mathcal{B}'$
  - Corresponds to shaving a codimension 2 face of $\mathcal{P}$
  - γ-vector remains nonnegative with each shave

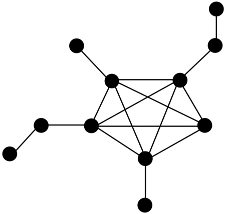
Isomorphisms $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$

	Non-extended	Extended ($\square$)
Polytopality	Yes	Yes for all $\mathcal{B}$
Gal's conjecture	Yes	Yes
When is $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$	Known for graphical $\mathcal{B}$ [MP17] Sufficient conditions for general $\mathcal{B}$ (these conditions also nec. for subclass of $\mathcal{B}$)	

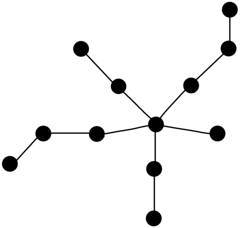
Isomorphisms $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$

Theorem (Manneville–Pilaud '17)

Let G, G' be undirected graphs such that $\mathcal{N}^{\square}(\mathcal{B}_G) \cong \mathcal{N}(\mathcal{B}_{G'})$. Then G is a **spider graph** and G' is the **corresponding octopus graph**.



Spider



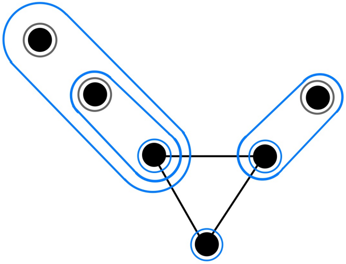
Octopus

$$\mathcal{N}^{\square}(\mathcal{B}_{K_n}) \cong \mathcal{N}(\mathcal{B}_{K_{1,n}}), \qquad \mathcal{N}^{\square}(\mathcal{B}_{P_{n-1}}) \cong \mathcal{N}(\mathcal{B}_{P_n})$$

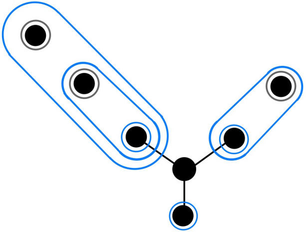
Isomorphisms $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$

Theorem (REU)

Let $\mathcal{B}$ be a **spider building set** and $\mathcal{B}'$ the corresponding **octopus building set**. Then $\mathcal{N}^{\square}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$.



Spider building set



Octopus building set

Building set elements are circled; additional elements formed by considering edges between building set elements circled in blue, along with middle vertex for octopus building set

Summary

	Non-extended	Extended ($\square$)
Polytopality	Yes	Yes for all $\mathcal{B}$
Gal's conjecture	Yes	Yes
When is $\mathcal{N}^\square(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$	Sufficient conditions for general $\mathcal{B}$	

Check out our preprint arXiv:1912.00273

Thank you!

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