A Proof of Grünbaum's Lower Bound Conjecture on general polytopes

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Face: intersection with a supporting hyperplane.

$k$-dimensional face: $k$-face.
Faces:

0-faces, 1-faces, ..., (d-1)-faces, d-face.

(vertices) (edges) (facets) polytope "itself"
**Faces:**
- 0-faces: vertices
- 1-faces: edges
- \( (d-1) \)-faces: facets
- d-face: polytope “itself”

**f-vector**

\[ f(P) = < f_0(P), f_1(P), \ldots, f_k(P), \ldots f_{d-1}(P) >\]

\# of k-faces
**Faces:**
- 0-faces (vertices)
- 1-faces (edges)
- \((d-1)\)-faces (facets)
- d-face

**f-vector**

\[ f(P) = < f_0(P), f_1(P), \ldots, f_k(P), \ldots, f_{d-1}(P) > \]

\# of \(k\)-faces

\[ f(P) = < 10, 15, 7 > \]

\[ P = \]
Vertex figure:
Vertex figure of $P$ at $v$: 

![Diagram of a vertex figure](image)
Vertex figure of $P$ at $v$:
Vertex figure of $P$ at $v$:

$P$

$P/v$:
Vertex figure of $P$ at $v$:

Prop.: $\{k$-faces of $P$ that contain $v\} \leftrightarrow \{\text{($k-1$)-faces of } P/\nu\}$
Main Questions: P: $d$-polytope with $n$ vertices.
Main Questions: \( P: \text{d-} \text{-polytope with } n \text{ vertices.} \)

- What is the MIN possible \( f_i(P) \)?
Main Questions: $P: d$-polytope with $n$ vertices.

- What is the MIN possible $f_i(P)$?

Moreover,

- Is there a polytope that has componentwise minimal $f$-vector?
Restricting to simplicial polytopes:
—Yes!
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- Yes!

- **LBT (Barnette 1973)**: Stacked polytopes have componentwise minimal $f$-vectors.
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— Yes!

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(and we know a lot more...
Restricting to simplicial polytopes:

—Yes!

- **LBT (Barnette 1973):** Stacked polytopes have componentwise minimal f-vectors.

  (and we know a lot more...)

- **UBT (McMullen 1970):** Cyclic polytopes have componentwise maximal f-vectors.

- **g-Thm (Billera-Lee, Stanley 1980 1980):** FULL characterization of f-vectors.
Restricting to simplicial polytopes:
— Yes!

- **LBT (Barnette 1973)**: Stacked polytopes have componentwise minimal f-vectors among simplicial polytopes.
  (and we know a lot more...)

- **UBT (McMullen 1970)**: Cyclic polytopes have componentwise maximal f-vectors among general polytopes.

- **g-Thm (Billera–Lee, Stanley 1980)**: FULL characterization of f-vectors of simplicial polytopes.
Is there a polytope that has componentwise minimal f-vector?

For general polytopes:

\( P: d\)-polytope over \( n \) vertices.
Is there a polytope that has componentwise minimal f-vector?

For general polytopes:

If \( n > 2d \) \( \Rightarrow \) NOT even a conjecture...
Is there a polytope that as componentwise minimal f-vector?

For general polytopes:

If \( n > 2d \) \( \Rightarrow \) NOT even a conjecture...

\( n \leq 2d \): Grünbaum's Conjecture (1967)
Grünbaum's Conjecture: \text{P: d-polytope over } \text{d+S vertices. (S \leq d)}

The number of k-faces of P is at least

\[(d+1) \binom{k+1}{d+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.\]
Grüenbaum’s Conjecture: \( P: \text{d-polytope over } d+S \text{ vertices.} (S \leq d) \)

The number of \( k \)-faces of \( P \) is at least

\[
\phi_k(d+S, d) := \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-S}{k+1}.
\]
Grüenbaum's Conjecture: \( P: d\)-polytope over \( d+S \) vertices. \((S \leq d)\)

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\]

Previous Results:

- Grüenbaum (1967): \( S = 2, 3, 4 \).
Grünbaum’s Conjecture: \( P: d \)-polytope over \( d+S \) vertices. \((S \leq d)\)

The number of \( k \)-faces of \( P \) is at least

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\]

Previous Results:

- Grünbaum (1967): \( S = 2, 3, 4 \).
- Pineda-Villavicencio, Ugon, Yost (2019): \( k = 1 \) (edge numbers)
Theorem 1 (X., 2020).

For all $d$ and $s \leq d$, let $P$ be a $d$-polytope with $d+s$ vertices, then

$$f_k(P) \geq \Phi_k(d+s, d) \text{ for every } k.$$
Key Prop.

$P$: $d$-polytope. For EVERY set of $m$ vertices ($m \leq d$)
$$\{v_1, v_2, \ldots, v_m\} \subseteq V(P),$$

$$\#\left\{ k\text{-faces of } P \text{ that contain some } v_i \right\} \geq \sum_{i=1}^{m} \binom{d-i+1}{k}.$$
The Proof \((\text{of Grünbaum's Conj.})\)

Induction on \(s\):

\[ P: d\text{-polytope}, f_0(P) = d + s \quad (s \leq d) \]
The Proof (of Grünbaum's Conj.)

Induction on $s$:

$P$: $d$-polytope, $f_0(P) = d + s$ ($s \leq d$)

Base case: $\checkmark$
**The Proof (of Grünbaum's Conj.)**

**Induction on** $s$:

- **Base case**: $\checkmark$
- **Inductive Step**: The statement holds for all $s' < s$ and all $d' \geq s' \implies$ Also hold for $s$ and all $d \geq s$
The Proof (of Grünbaum's Conj.)

Induction on $s$: $P$: $d$-polytope, $f_0(P) = d + s$ ($s \leq d$)

Base case: ✓

Inductive Step: The statement holds for all $s' < s$ and all $d' \geq s'$ ⇒ Also hold for $s$ and all $d \geq s$

- Pick a facet $F$ with $f_0(F) = d + s - m$, $m > 1$.
- $\{v_1, \ldots, v_m\} = V(P) - V(F)$. 
The Proof (cont.)

- $k$-faces of $P$
The Proof (cont.)

- $k$-faces of $P$ $\rightarrow$ $k$-faces of $F$.

$\Rightarrow$ Containing some $v_i$. 
The Proof (cont.)

- $k$-faces of $P \rightarrow k$-faces of $F$ (inductive hyp.)

$\Rightarrow$ Containing some $\nu_i$
The Proof (cont.)

\[ k\text{-faces of } P \rightarrow k\text{-faces of } F \quad \text{(inductive hyp.)} \]

\[ \rightarrow \text{containing some } v_i \quad \text{(Key Prop.)} \]
The Proof (cont.)

- $k$-faces of $P$ $\rightarrow$ $k$-faces of $F$ (inductive hyp.)

$\Rightarrow$ containing some $\nu_i$ (Key Prop.)

$$f_k(P) \geq \phi_k(d+s-m, d-1) + \sum_{i=1}^{m} \binom{d-i+1}{k}$$
The Proof (cont.)

- $k$-faces of $P \rightarrow k$-faces of $F$ (inductive hyp.)

  \[ f_k(P) \geq \phi_k(d+s-m, d-1) + \sum_{i=1}^{m} \binom{d-i+1}{k} \]

  
  \[ \quad \text{(rearrangement)} \]

  \[ = \phi_k(d+s, d) + \sum_{i=3}^{m} \left[ \binom{d-i+1}{k} - \binom{d-i+1-(s-m)}{k} \right] \geq \phi_k(d+s, d). \]
The Proof (cont.)

- If there exists NO facet with $d+s-m$ vertices with $m \geq 1$, 

![Diagram](image.png)
The Proof (cont.)

- If there exists NO facet with \(d+s-m\) vertices with \(m>1\), then every facet has exactly \(d+s-1\) vertices. Hence \(P\) is a \(d\)-simplex.
Treatment of Equality
Treatment of Equality

Which polytope $P$ has $f_k(P) = \Phi_k(d+s, d)$ for ALL $k$'s?
Treatment of Equality

Which polytope $P$ has

$$f_k(P) = \Phi_k(d+s, d)$$

for \{ALL \ k's\}?

\{SOME\}
Pyramid:

Apex

Basis
Pyramid:

- Apex
- Basis

k-fold pyramid
Pyramid:

A point $x \in \mathbb{R}^d$ beyond a facet:
Pyramid:

A point $x \in \mathbb{R}^d$ beyond a facet $F$.
Pyramid:

A point $x \in \mathbb{R}^d$ beyond a facet $F$: beyond 1 facet
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A point $x \in \mathbb{R}^d$ beyond a facet $F$: beyond 1 facet
Pyramid:

A point \( x \in \mathbb{R}^d \) beyond a facet \( F \):

- beyond 2 facets
- Pyramid:

- A point $x \in \mathbb{R}^d$ beyond a facet $F$:

- Dual polytope
- Pyramid:
- A point $x \in \mathbb{R}^d$ beyond a facet $F$
- Dual polytope $P^*$
- Pyramid:
- A point $x \in \mathbb{R}^d$ beyond a facet $F$
- Dual polytope $P^*$
Notation: \((a \geq 0)\)

\(\Gamma^a\)

\(\alpha\)-simplex
Notation: \((a \geq 0)\)

\[ T^a \rightarrow T_m^a \]

\( a \)-simplex

Put a new vertex beyond \( m \) facets

Simplicial \( a \)-polytope
Notation: \((\alpha \geq 0)\)

- \(\tau^\alpha\): \(\alpha\)-simplex
  - \(m\) facets
  - Put a new vertex beyond

- \(T_m^\alpha\): simplicial \(\alpha\)-polytope
  - \((d-a)\)-fold pyramid

- \(T_m^{d-d-a}\): \(d\)-polytope
Notation: \((a \geq 0)\)

- \(\tau^a\): \(a\)-simplex
- \(T^a_m\): \(a\)-polytope
- \(T^{d-d-a}_m\): \(d\)-polytope
- \(\tau^a_{m}\): \((d-a)\)-fold pyramid

Equivalently,

\[ T^a_m = T^m_m \oplus T^{a-m}_m \]

\[ T^{d-d-a}_m = T^{d-a-1}_m \times (T^m_m \oplus T^{a-m}_m) \]
**Lemmas (Grünbaum, 1967)**

1. **Lem. 1**  \( T_m^d = T_{d-m}^d \)
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$\Delta^3$
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- Lem. 1: $T^d_m = T^d_{d-m}$. 

\[
T_1^3 \sim T_2^3
\]
Lemmas (Grünbaum, 1967)

- **Lem. 1:** \( T_m^d = T_{d-m}^d \).

- **Lem. 2:** Every simplicial \( d \)-polytope with \( d+2 \) vertices is \( T_m^d \) for some \( m \) (\( 1 \leq m \leq d-1 \)).
Lemmas (Grünbaum, 1967)

- Lem. 1: $T_m^d = T_{d-m}^d$. 

- Lem. 2: Every simplicial $d$-polytope with $d+2$ vertices is $T_m^d$ for some $m$ ($1 \leq m \leq \lfloor \frac{d}{2} \rfloor$).
Lemmas (Grünbaum, 1967)

Lemma 3: For $0 \leq k \leq d-1$, $2 \leq a \leq d$, and $1 \leq m \leq \left\lfloor \frac{a}{2} \right\rfloor$,

$$f_k(T_m^{d,d-a}) = \binom{d+2}{d-k+1} - \binom{d-a+m-1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.$$
Lemmas (Grünbaum, 1967)

- **Lem. 3:** For $0 \leq k \leq d-1$, $2 \leq a \leq d$, and $1 \leq m \leq \lfloor \frac{a}{2} \rfloor$,
  \[ f_k(T_{m}^{d,d-a}) = {d+2 \choose d-k+1} - {d-a+m-1 \choose d-k+1} - {d-m+1 \choose d-k+1} + {d-a+1 \choose d-k+1}. \]

- **Corollary:** $f_k((T_{1}^{d,d-s})^*) = \Phi_k(d+s, d)$. 

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Lemma 3: For $0 \leq k \leq d-1$, $2 \leq a \leq d$, and $1 \leq m \leq \lfloor \frac{a}{2} \rfloor$,

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Corollary: $f_k((T_d^{d,d-s})^*) = \Phi_k(d+s, d)$. 
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Lem. 3: For $0 \leq k \leq d-1$, $2 \leq a \leq d$, and $1 \leq m \leq \left\lfloor \frac{a}{2} \right\rfloor$,

$$f_k(T_m^{d,d-a}) = \binom{d+2}{d-k+1} - \binom{d-a+m-1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.$$  

Corollary: $f_k((T_1^{d,d-s})^*) = \Phi_k(d+s, d)$.  

(Any other minimizers?)
Corollaries (of proof of Thm. 1)

If $f_{k}(P) = \Phi_{k}(d+s, d)$ for some $1 \leq k \leq d-2$, then

1. Each facet of $P$ has $d$, $d+s-2$, or $d+s-1$ vertices.
2. Every non-apex vertex is simple.
3. $P$ has $d+2$ facets.
Theorem 2 (X. 2020)

Let $P$ be a $d$-polytope with $d+s$ vertices where $s \leq d$. If $f_k(P) = \Phi_k(d+s, d)$ for some $k$ with $1 \leq k \leq d-2$, then $P = (T^d_{1,d-s})^*$. 
Theorem 2 (X. 2020)

Let $P$ be a $d$-polytope with $d+s$ vertices where $s \leq d$. If $f_k(P) = \phi_k(d+s, d)$ for some $k$ with $1 \leq k \leq d-2$, then $P = (T^d_{1, d-s})^*$.

Remark (the case of $k = d-1$):
**Theorem 2** (X. 2020)

Let $P$ be a $d$-polytope with $d+s$ vertices where $s \leq d$. If $f_k(P) = \phi_k(d+s,d)$ for some $k$ with $1 \leq k \leq d-2$, then $P = (T^d,d-s)^*$. 

**Remark** (the case of $k = d-1$): Let $2 \leq s \leq d$. If $f_0(P) = d+s$, and $f_{d-1}(P) = d+2$, then $P = (T^{d,d-a}_m)^*$ for some $2 \leq a \leq d$, $1 \leq m \leq \left\lfloor \frac{a}{2} \right\rfloor$, and $m(a-m) = s-1$. 
That’s it!

Thank you!
What if \( f_0 > 2d \)?
What if $f_0 > 2d$?

Componentwise MIN. $f$-vector might NOT exist.

$f_0 = 2d + 1$:

- $P_1 = \text{Stack } (T_{1,d}, d-2)^*$:
  - $P_2 = (T_{2,d}, \frac{d}{2}-2)^*$.

$d = 3$: the "Pentasm"
\[ P_1 = \text{Stack} \left( T_i^d, d-2 \right)^* \]
\[ P_2 = \left( T_2^d, \frac{a}{2} - 2 \right)^* \]

Pineda-Villavicencio, Ugon, Yost (2019):

For \( d \geq 5 \), \( \min f_i = f_i(P_i) < f_i(P_2) \) (unique!)

But...
- \( P_1 = \text{Stack}\left( T_i^d, d-2 \right)^* \).

- \( P_2 = \left( T_2^d, \frac{d}{2}-2 \right)^* \).

Pineda-Villavicencio, Ugon, Yost (2019):

For \( d \geq 5 \), \( \min f_i = f_i(P_1) < f_i(P_2) \).

But...

For \( d \) even, \( \min f_{d-1} = f_{d-1}(P_2) < f_{d-1}(P_1) \).
\[ P_1 = \text{Stack} \left( T_1^{d, d-2} \right)^* \]

\[ P_2 = \left( T_2^{d, \frac{d}{2}-2} \right)^* \]

Pineda-Villavicencio, Ugon, Yost (2019):

For \( d \geq 5 \), \( \min f_i = f_i (P_1) < f_i (P_2) \)

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For \( d \text{ even} \), \( \min f_{d-1} = f_{d-1} (P_2) < f_{d-1} (P_1) \)
\[ P_1 = \text{Stack}(T_1^{d,d-2})^* \]
\[ P_2 = (T_2^d, \frac{d}{2}-2)^* \]

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But...

For \( d \) even, \( \min f_{d-1} = f_{d-1}(P_2) < f_{d-1}(P_1) \).
Thanks again!