A Proof of Grünbaum's Lower Bound Conjecture on general polytopes

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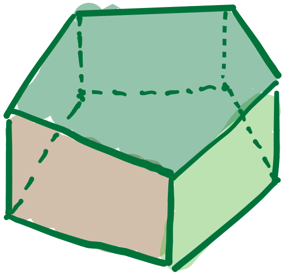

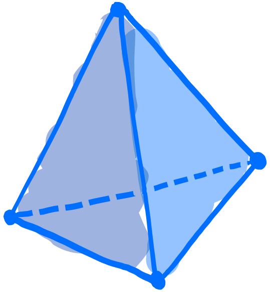

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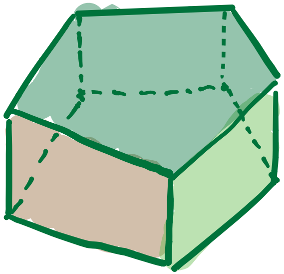

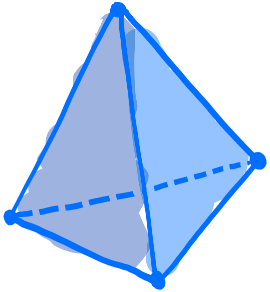

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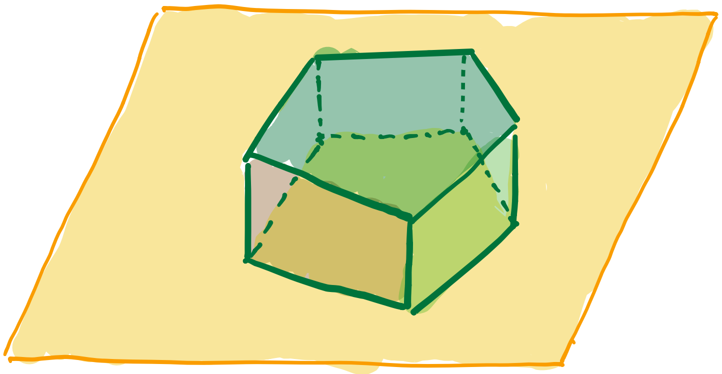

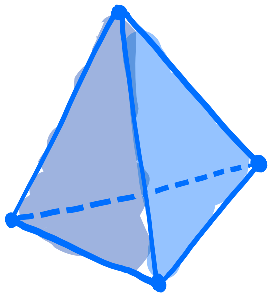

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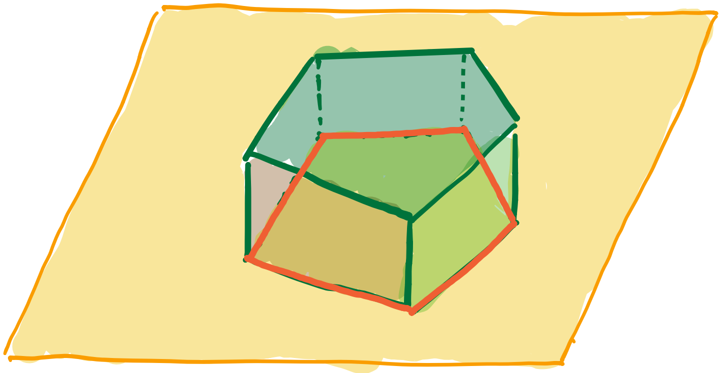

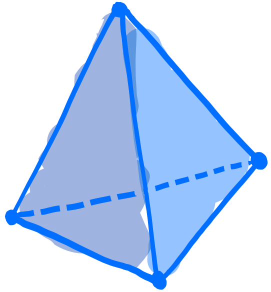

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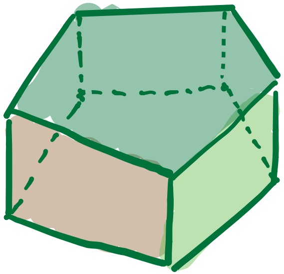

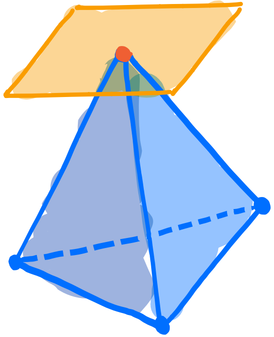

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k-dimensional face : k-face

- **Faces:**

  0-faces (vertices), 1-faces (edges), ..., (d-1)-faces (facets), d-face: polytope "itself".

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f-vector

$$f(P) = \langle f_0(P), f_1(P), \cdots, f_k(P), \cdots f_{d-1}(P) \rangle$$

\# of k-faces

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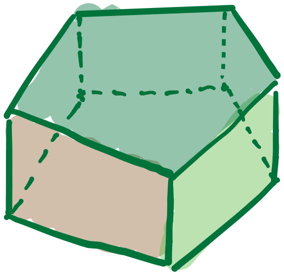
f_k: # of k-faces

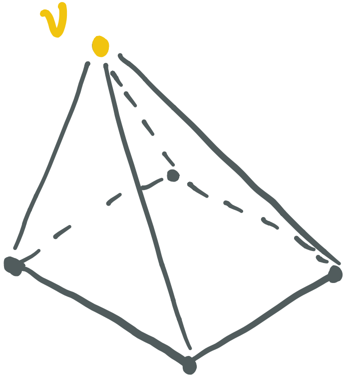
$$f(P) = \langle 10, 15, 7 \rangle$$

P = 

- Vertex figure:

- Vertex figure of P at v:

P



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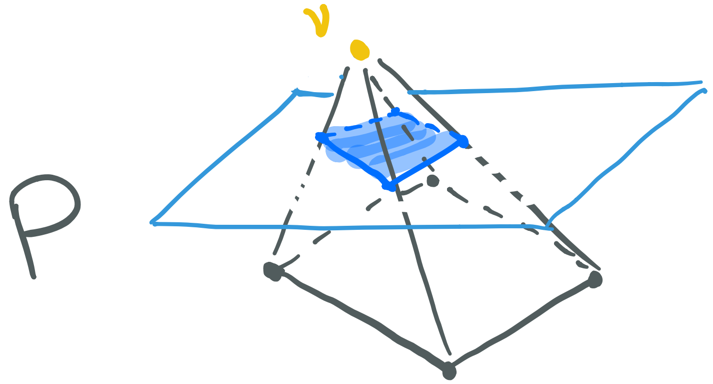

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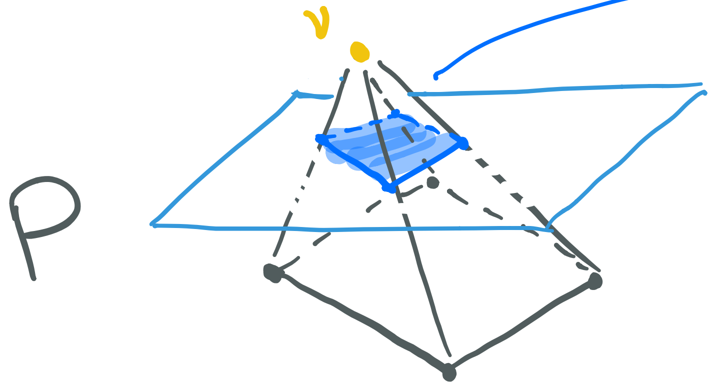

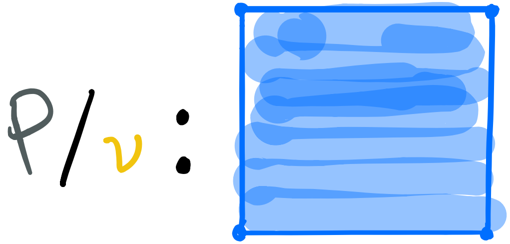

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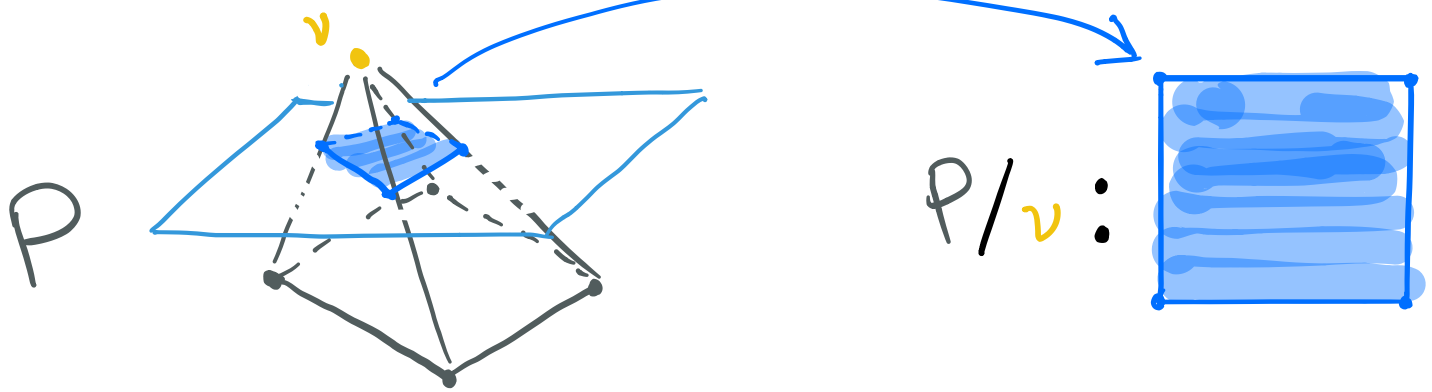

- Prop.: $\{k\text{-faces of } P \text{ that contain } v\} \longleftrightarrow \{(k-1)\text{-faces of } P/v\}$

Main Questions:

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Moreover,

- Is there a polytope that has componentwise minimal f-vector?

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Restricting to simplicial polytopes:

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- LBT (Barnette 1973): Stacked polytopes have componentwise minimal f-vectors among simplicial polytopes.

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- UBT (McMullen 1970): Cyclic polytopes have componentwise maximal f-vectors among general polytopes.
- g-Thm (Billera-Lee, Stanley): FULL characterization of f-vectors of simplicial polytopes.

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$n \leq 2d$: Grünbaum's Conjecture (1967)

Grünbaum's Conjecture:

P: d-polytope over
d+s vertices. ($s \le d$)

The number of k-faces of P is at least

$$\binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.$$

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Previous Results:

- Grünbaum (1967): $s = 2, 3, 4$.
- Pineda-Villavicencio, Ugon, Yost (2019): $k=1$ (edge numbers)

- 2020

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Theorem 1 (X., 2020).

For all d and $s \leq d$, let P be a d-polytope with $d+s$ vertices, then

$$f_k(P) \geqslant \phi_k(d+s, d) \quad \text{for every } k.$$

Key Prop.

P: d-polytope. For **EVERY** set of m vertices ($m \le d$) $\{v_1, v_2, \cdots, v_m\} \subseteq V(P)$,

$$\#\left\{\begin{array}{c}k\text{-faces of } P \\ \text{that contain some } v_i\end{array}\right\} \geq \sum_{i=1}^{m} \binom{d-i+1}{k}.$$

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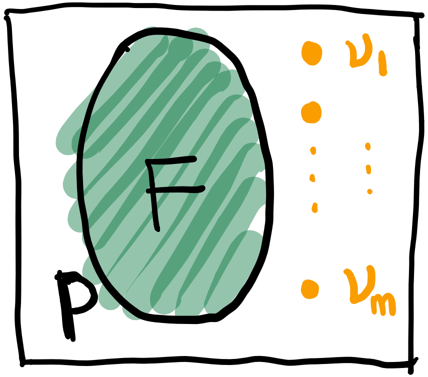
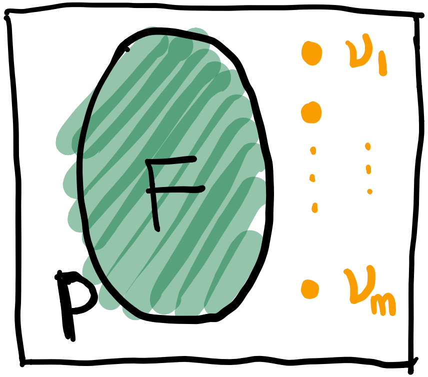
- Pick a facet F with $f_0(F) = d+s-m$, $m > 1$.
- $\{v_1, \cdots, v_m\} = V(P) - V(F)$.



The Proof (cont.)

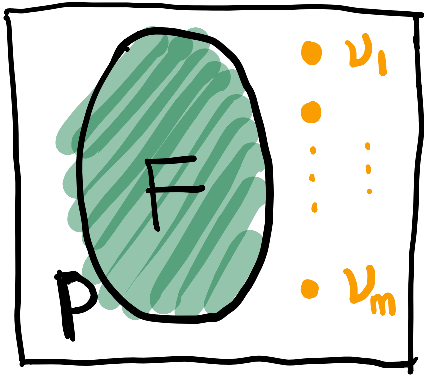
- k-faces of P



The Proof (cont.)

- k-faces of $P \longrightarrow$ k-faces of F.

  $\rightarrow$ containing some v_i.



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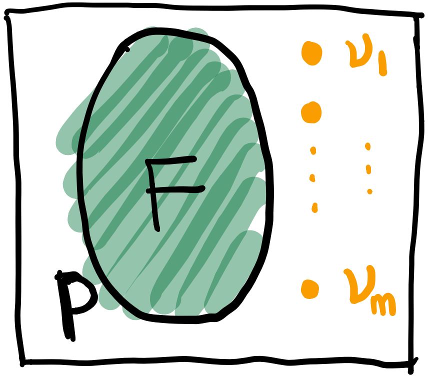

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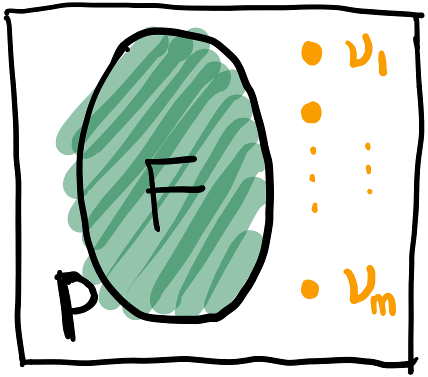

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$$f_k(P) \geq \phi_k(d+s-m,\, d-1) + \sum_{i=1}^{m} \binom{d-i+1}{k}$$

The Proof (cont.)

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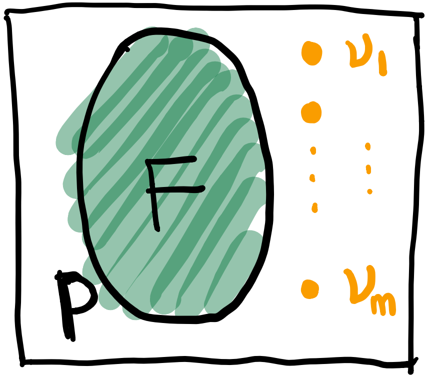
... (rearrangement)

$$= \phi_k(d+s, d) + \sum_{i=3}^{m} \underbrace{\left[\binom{d-i+1}{k} - \binom{d-i+1-(s-m)}{k} \right]}_{\geq 0}$$

$$\geq \phi_k(d+s, d).$$

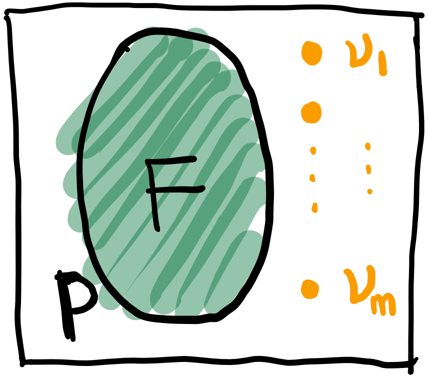
The Proof (cont.)

- If there exists NO facet with $d+s-m$ vertices with $m>1$,



The Proof (cont.)

- If there exists NO facet with $d+s-m$ vertices with $m>1$, then every facet has exactly $d+s-1$ vertices. Hence P is a d-simplex. ∎



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Which polytope P has

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for ALL k's?

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for $\begin{cases} \text{ALL} \\ \text{SOME} \end{cases}$ k's?

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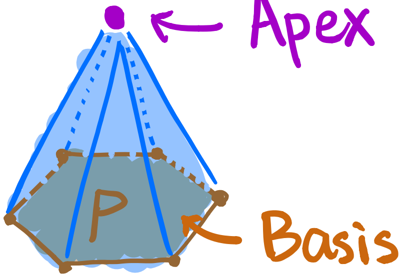

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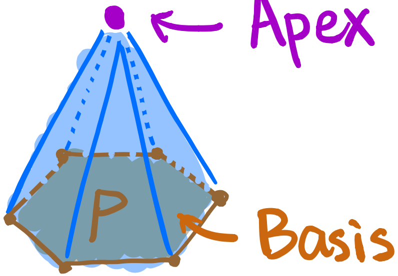

- k-fold pyramid

- Pyramid:

- A point $x \in \mathbb{R}^d$ beyond a facet:

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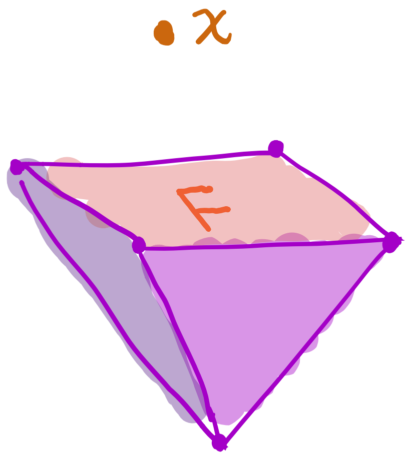

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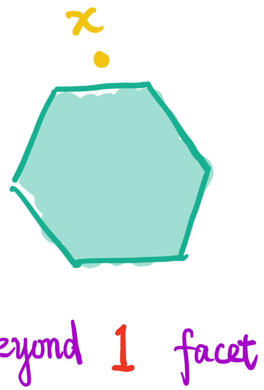

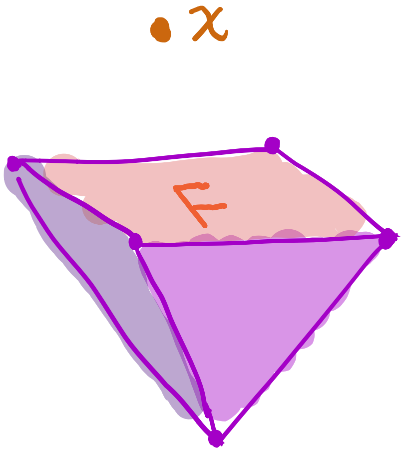

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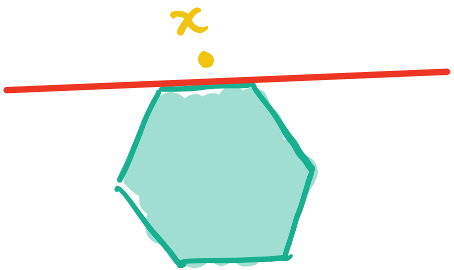
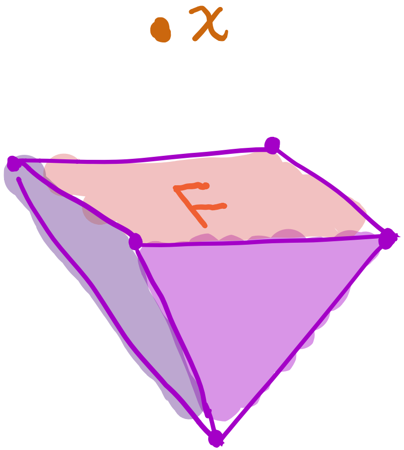


beyond 1 facet



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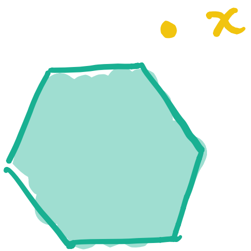
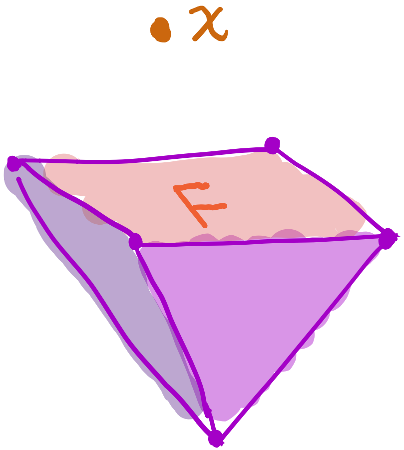


beyond 2 facets



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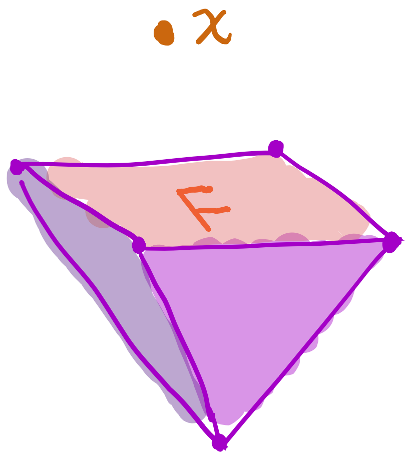

- Dual polytope

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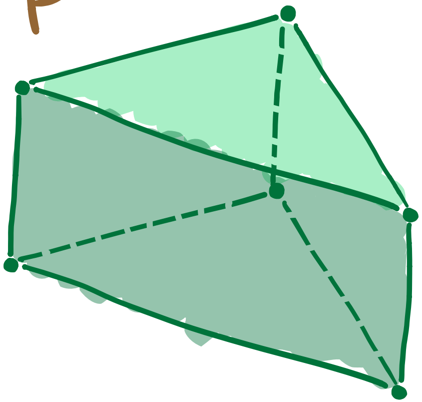

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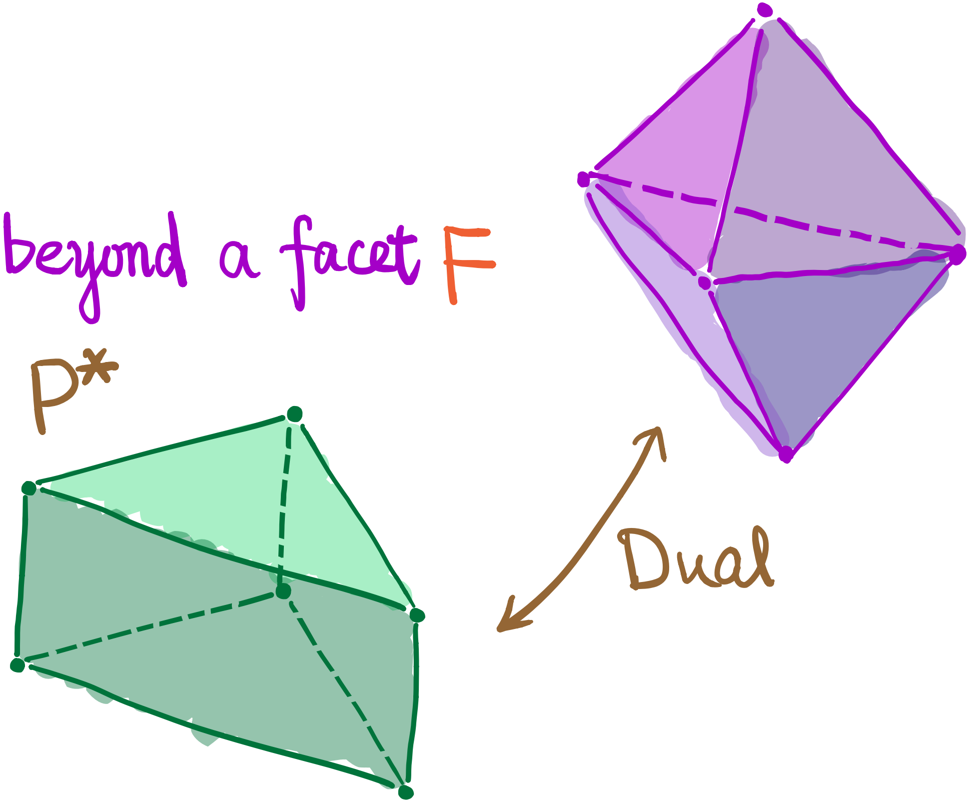

Notation : $(a \geq 0)$

T^a

a-simplex

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$$T^a \xrightarrow[\text{m facets}]{\text{put a new vertex beyond}} T^a_m$$

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simplicial a-polytope

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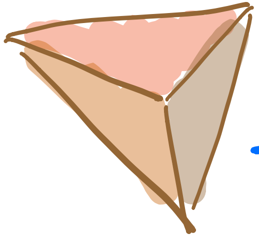
Equivalently, $T^a_m = T^m \oplus T^{a-m}$

$$T^{d,d-a}_m = T^{d-a-1} * (T^m \oplus T^{a-m})$$

Lemmas (Grünbaum, 1967)

- Lem. 1 $T^d_m = T^d_{d-m}$.



T^3_1

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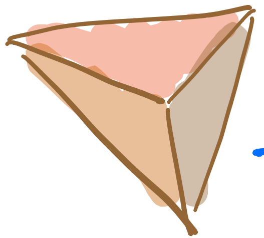

T_1^3

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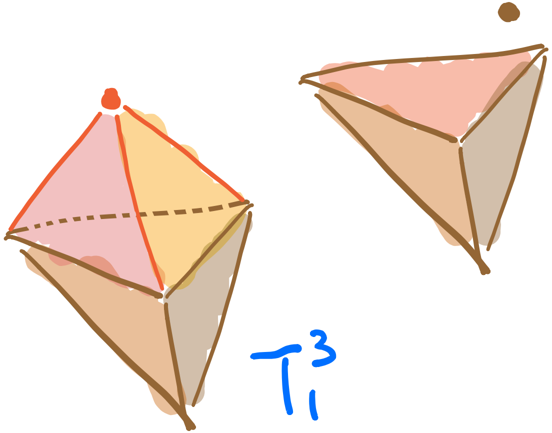

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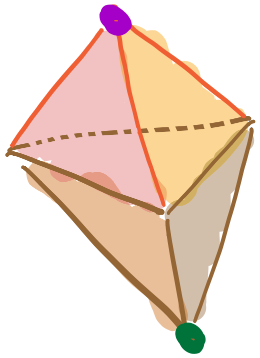
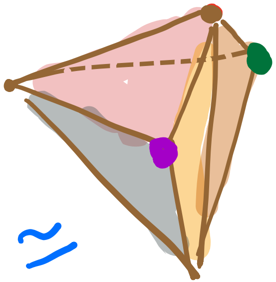


T^3_1 $\simeq$ T^3_2



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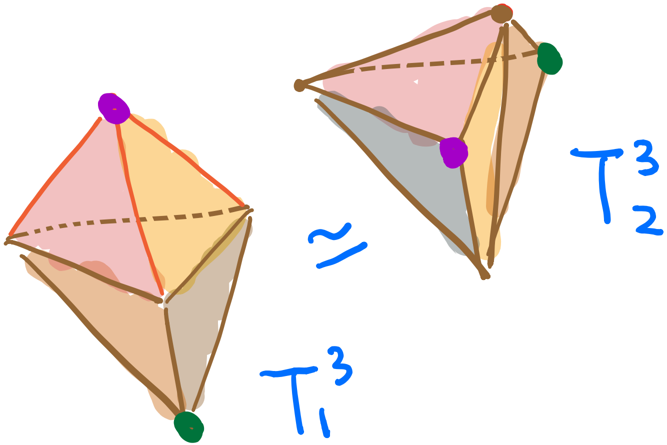

- Lem. 2: Every simplicial d-polytope with d+2 vertices is T_m^d for some m $(1 \leq m \leq d-1)$.

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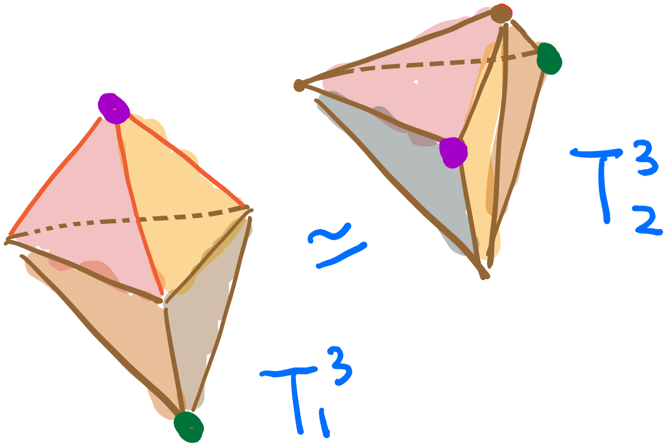
- Lem. 1: $T^d_m = T^d_{d-m}$.



- Lem. 2: Every simplicial d-polytope with d+2 vertices is T^d_m for some m $\left(1 \le m \le \lfloor \frac{d}{2} \rfloor\right)$.

Lemmas (Grünbaum, 1967)

- Lem. 3: For $0 \leq k \leq d-1$, $2 \leq a \leq d$, and $1 \leq m \leq \lfloor \frac{a}{2} \rfloor$,

$$f_k(T_m^{d,d-a}) = \binom{d+2}{d-k+1} - \binom{d-a+m-1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.$$

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(Any other minimizers ?)

Corollaries (of proof of Thm. 1)

If $f_k(P) = \phi_k(d+s, d)$ for some $1 \le k \le d-2$, then

1. Each facet of P has d, $d+s-2$, or $d+s-1$ vertices.
2. Every non-apex vertex is simple.
3. P has $d+2$ facets.

Theorem 2 (X. 2020)

Let P be a d-polytope with $d+s$ vertices where $s \leq d$.

If $f_k(P) = \phi_k(d+s, d)$ for some k with $1 \leq k \leq d-2$,

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Remark (the case of $k = d-1$): Let $2 \leq s \leq d$.

If $f_0(P) = d+s$, and $f_{d-1}(P) = d+2$, then $P = (T_m^{d,d-a})^*$
for some $2 \leq a \leq d$, $1 \leq m \leq \lfloor \frac{a}{2} \rfloor$, and $m(a-m) = s-1$.

That's it!

Thank you!

- What if $f_0 > 2d$?

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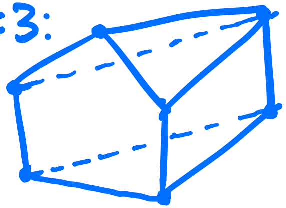
Componentwise MIN. f-vector might NOT exist.

$f_0 = 2d+1$:



- $P_1 = \mathrm{Stack}\,(T_1^{d,\,d-2})^*$: the "Pentasm"
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Pineda-Villavicencio, Ugon, Yost (2019):

For $d \geq 5$, $\quad \mathrm{MIN}\, f_1 = f_1(P_1) < f_1(P_2)$

(unique!)

But...

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- $P_2 = (T_2^{d,\,\frac{d}{2}-2})^*$.

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For d even, $\mathrm{MIN}\, f_{d-1} = f_{d-1}(P_2) < f_{d-1}(P_1)$.

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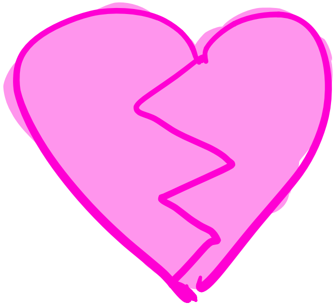

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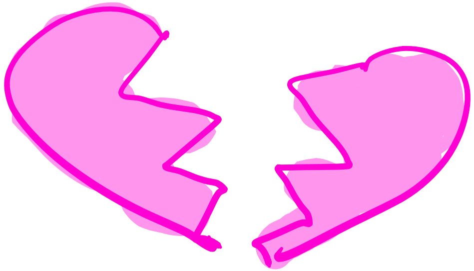

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Thanks again!