New combinatorial models for the Genocchi and median Genocchi numbers.

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Joint work with Alex Lazar

The Genocchi numbers

n	1	2	3	4	5	6
g_n	1	1	3	17	155	2073
h_n	2	8	56	608	9440	198272

$$\sum_{n\geq 1} g_n \, \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}$$

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Barsky-Dumont (1979):

$$\sum_{n\geq 1} g_n x^n = \sum_{n\geq 1} \frac{(n-1)!n!\, x^n}{\prod_{k=1}^{n}(1+k^2 x)}$$

$$\sum_{n\geq 1} h_n x^n = \sum_{n\geq 1} \frac{n!(n+1)!x^n}{\prod_{k=1}^{n}(1+k(k+1)x)}$$

Combinatorial definition - Dumont 1974

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Genocchi numbers:

$$g_n = |\{\sigma \in \mathfrak{S}_{2n-2} : i \leq \sigma(i) \text{ if } i \text{ is odd; } \; i > \sigma(i) \text{ if } i \text{ is even}\}|.$$

These are called Dumont permutations.

$$g_3 = |\{(1,2)(3,4),\ (1,3,4,2),\ (1,4,2)(3)\}| = 3.$$

median Genocchi numbers:

$$h_n = |\{\sigma \in \mathfrak{S}_{2n+2} : i < \sigma(i) \text{ if } i \text{ is odd; } \; i > \sigma(i) \text{ if } i \text{ is even}\}|.$$

These are called Dumont derangements.

$$h_1 = |\{(1,2)(3,4),\ (1,3,4,2)\}| = 2.$$

New permutation models

$\sigma \in \mathfrak{S}_{2n}$ is a D-permutation if $i \leq \sigma(i)$ whenever i is odd and $i \geq \sigma(i)$ whenever i is even.

$\{(1,2)(3,4),\ (1,3,4,2),\ (1,4,2)(3),\ (1,2)(3)(4),$
$(1,4)(2)(3),\ (3,4)(1)(2),\ (1,3,4)(2),\ (1)(2)(3)(4)\}$

$\mathcal{DC}_{2n} = \{D\text{-cycles on } [2n]\}, \qquad \mathcal{D}_{2n} = \{D\text{-permutations on } [2n]\}.$

$\mathcal{DC}_{2n} \subseteq \{\text{Dumont derange. on } [2n]\} \subseteq \{\text{Dumont perm. on } [2n]\} \subseteq \mathcal{D}_{2n}.$

h_{n-1} $\qquad$ g_{n+1}

New permutation models

$\sigma \in \mathfrak{S}_{2n}$ is a E-permutation if $i > \sigma(i)$ implies i is even and $\sigma(i)$ is odd.

$\{(1,2)(3,4),\ (1,2,4)(3),\ (1,3,4)(2),\ (1,2)(3)(4),$
$(1,4)(2)(3),\ (3,4)(1)(2),\ (1,2,3,4),\ (1)(2)(3)(4)\}$

$$\mathcal{EC}_{2n} = \{E\text{-cycles on } [2n]\}, \qquad \mathcal{E}_{2n} = \{E\text{-permutations on } [2n]\}.$$

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Theorem (Lazar, W.)

$h_n = |\mathcal{D}_{2n}| = |\mathcal{E}_{2n}|$

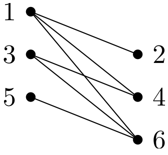
$g_n = |\mathcal{DC}_{2n}|$

Conjecture (Lazar, W.)

$g_n = |\mathcal{EC}_{2n}|$

Chromatic polynomial

Let Γ_{2n} be the bipartite graph on vertex set $\{1, 3, \ldots, 2n-1\} \sqcup \{2, 4, \ldots, 2n\}$ with an edge between $2i-1$ and $2j$ for all $i \leq j$.



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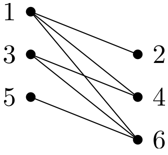

Let $\chi_{\Gamma_{2n}}(t)$ be the chromatic polynomial of $\Gamma_{2n}(t)$.

n	$t^{-1}\chi_{\Gamma_{2n}}(t)$	$t=-1$	$t=0$
1	$t-1$	-2	-1
2	t^3-3t^2+3t-1	-8	-1
3	$t^5-6t^4+15t^3-19t^2+12t-3$	-56	-3
4	$t^7-10t^6+45t^5-115t^4+177t^3-162t^2+81t-17$	-608	-17

Generating function for the chromatic polynomial

Theorem (Lazar, W.)

$$\sum_{n\geq 1} \chi_{\Gamma_{2n}}(t)\, z^n = \sum_{n\geq 1} \frac{(t)_n (t-1)_n\, z^n}{\prod_{k=1}^{n}(1-k(t-k)z)}.$$

where $(a)_n$ denotes the falling factorial $a(a-1)\cdots(a-n+1)$.

Multiply by $-t^{-1}$ and set $t=0$. We get the Barsky-Dumont generating function for g_n:

$$\sum_{n\geq 1} g_n z^n = \sum_{n\geq 1} \frac{(n-1)!n!\, z^n}{\prod_{k=1}^{n}(1+k^2 z)}.$$

Multiply by $-t^{-1}$ and set $t=-1$. We get the Barsky-Dumont generating function for h_n:

$$\sum_{n\geq 1} h_n z^n = \sum_{n\geq 1} \frac{n!(n+1)!\, z^n}{\prod_{k=1}^{n}(1+k(k+1)z)}.$$

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where $(a)_n$ *denotes the falling factorial* $a(a-1)\cdots(a-n+1)$.

Corollary:

$$-(t)^{-1}\chi_{\Gamma_{2n}}(t) = \begin{cases} h_n & \text{if } t=-1 \\ g_n & \text{if } t=0 \end{cases}.$$

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Proof steps

1. use the Rota-Whitney NBC theorem to determine the coefficients of $\chi_{\Gamma_{2n}}(t)$ by counting a certain set $\mathcal{F}_{2n}$ of forests .
2. construct a bijection from $\mathcal{F}_{2n}$ to $\mathcal{D}_{2n}$
3. construct a bijection from $\mathcal{D}_{2n}$ to a certain set of "surjective staircases"
4. use a generating function of Randrianarivony-Zeng (1996) for an enumerator of surjective staircases with multiple statistics.

D-permutations

The first two steps yield,

$$\chi_{\Gamma_{2n}}(t) = \sum_{\sigma \in \mathcal{D}_{2n}} (-t)^{cyc(\sigma)},$$

where $cyc(\sigma)$ denotes the number of cycles of σ.

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- Since g_n is obtained by setting $t = 0$ in

$$-(t)^{-1}\chi_{\Gamma_{2n}}(t) = \sum_{\sigma \in \mathcal{D}_{2n}} (-t)^{cyc(\sigma)-1},$$

  we get $g_n = |\mathcal{DC}_{2n}|$
- Since h_n is obtained by setting $t = -1$ in

$$-(t)^{-1}\chi_{\Gamma_{2n}}(t) = \sum_{\sigma \in \mathcal{D}_{2n}} (-t)^{cyc(\sigma)-1},$$

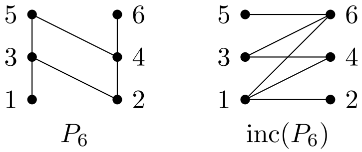
  we get $h_n = |\mathcal{D}_{2n}|$

E-permutations

Recall Γ_{2n} is the bipartite graph on vertex set $\{1, 3, \ldots, 2n-1\} \sqcup \{2, 4, \ldots, 2n\}$ with an edge between $2i-1$ and $2j$ for all $i \leq j$.

Observation: Γ_{2n} is the incomparability graph of the poset P_{2n} on $[2n]$ with order relation given by $x \leq_{P_{2n}} y$ if:

- $x \leq y$ and $x \equiv y \bmod 2$
- $x < y$, x is even, and y is odd.



A result of Chung and Graham

A permutation σ of the vertices of a poset P has a P-drop at x if $x >_P \sigma(x)$.
Example: The cycle (532164) has P_6-drops at 5, 3, 6 only. Not 2

Chung-Graham (1995): For any finite poset P,

$$\chi_{\mathrm{inc}(P)}(t) = \sum_{k=0}^{|P|-1} d(P,k)\binom{t+k}{|P|},$$

where $d(P,k)$ is the number of permutations of P with exactly k P-drops.

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$$\chi_{\mathrm{inc}(P)}(-1) = \sum_{k=0}^{|P|-1} d(P,k)\binom{k-1}{|P|} = d(P,0).$$

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$$\chi_{\mathrm{inc}(P)}(-1) = \sum_{k=0}^{|P|-1} d(P,k) \binom{k-1}{|P|} = d(P,0).$$

A permutation in $\sigma \in \mathfrak{S}_{2n}$ has no P_{2n}-drops if for all $i \in [2n]$, $i > \sigma(i)$ implies i is even and $\sigma(i)$ is odd.

E-permutations

Putting all this together, we have

$$h_n = \chi_{\Gamma_{2n}}(-1) = |\mathcal{E}_{2n}|$$

Conjecture

The number of D-permutations on $[2n]$ with k cycles equals the number of E-permutations on $[2n]$ with k cycles for all k. Consequently

$$g_n = |\mathcal{EC}_{2n}|.$$

We have verified this by computer for $n \leq 6$.

Byproduct: expansion in powers of 2

Theorem (Lazar-W.)

$$h_n = \sum_{j=1}^{n-1} h_{n-1,j}\, 2^{j+1}$$

$$g_n = \sum_{j=0}^{n-2} g_{n-2,j}\, 2^{j}$$

where

- $h_{n,j}$ *is the number of D-permutations on* $[2n]$ *with exactly* j *cycles that are not even fixed points,*
- $g_{n,j}$ *is the number of D-permutations on* $[2n]$ *with exactly* j *cycles that are not fixed points*

Sundaram (1995) also has an expansion for g_n in powers of 2.

Some other geometric models and refinements

- Sundaram (1995): Möbius invariant of poset of partitions of $[2n]$ with an even number of blocks equals $(2n-1)!g_n$
- Feigin (2011): Poincaré polynomial of a certain degenerate flag variety refines the normalized median Genocchi numbers $\frac{h_n}{2^{n-1}}$.
- Hetyei (2017): Number of regions in the homogenized Linial arrangement equals h_n.
- Lazar,W. (2019): Type B and Dowling analogs: $h_n(m)$ and $g_n(m)$
- Lazar (2020): Generalization to other Ferrers graphs