Decompositions of Ehrhart $h^*$-polynomials for rational polytopes

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Let \( P \) be a rational \( d \)-polytope in \( \mathbb{R}^d \), i.e., convex hull of finitely many points in \( \mathbb{Q}^d \).

For a positive integer \( t \), let \( L_P(t) \) denote the number of integer lattice points in \( tP \).

**Theorem:** (Ehrhart 1962) Given a rational polytope \( P \), the counting function \( L_P(t) := |tP \cap \mathbb{Z}^d| \) is a quasipolynomial of the form

\[
\text{vol}(P) t^d + k_{d-1}(t)t^{d-1} + \cdots + k_1(t)t + k_0(t),
\]

where \( k_0(t), k_1(t), \ldots, k_{d-1}(t) \) are periodic functions in \( t \).

We call \( L_P(t) \) the Ehrhart quasipolynomial of \( P \), and each period of \( k_0(t), k_1(t), \ldots, k_{d-1}(t) \) divides the denominator \( q \) of \( P \), which is the least common multiple of all its vertex coordinate denominators.
Let $P$ to be a rational $d$-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$. 

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where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in $t$. We call $L_P(t)$ the Ehrhart quasipolynomial of $P$, and each period of $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ divides the denominator $q$ of $P$, which is the least common multiple of all its vertex coordinate denominators.
Let $P$ to be a **rational** $d$-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$.

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Main Characters: (Rational) Polytopes!

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Ehrhart Quasipolynomials

A quasipolynomial $L_P(t)$ is a function $Z \to \mathbb{R}$ of the form

$$L_P(t) = k_d(t) t^d + \cdots + k_1(t) t + k_0(t),$$

where $k_0, \ldots, k_d$ are periodic functions in the integer variable $t$.

Alternatively, for a quasipolynomial, there exist a positive integer $q$ and polynomials $f_0, \ldots, f_{p-1}$, such that

$$L_P(t) = \begin{cases} f_0(t) & \text{if } t \equiv 0 \mod q \\ f_1(t) & \text{if } t \equiv 1 \mod q \\ \vdots & \\ f_{p-1}(t) & \text{if } t \equiv q-1 \mod q \end{cases}.$$
A quasipolynomial \( L_P(t) \) is a function \( \mathbb{Z} \rightarrow \mathbb{R} \) of the form

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  f_{p-1}(t) & \text{if } t \equiv q - 1 \mod q.
\end{cases}$$
The Ehrhart series is the rational generating function

$$E_{\text{hr}}(P; z) := \sum_{t \geq 0} L_P(t) z^t = h^\ast(P; z) (1 - z^q)^{-1},$$

where $h^\ast(P; z)$ is a polynomial of degree less than $q(d + 1)$ called the $h^\ast$-polynomial of $P$. 
The *Ehrhart series* is the rational generating function

\[ \text{Ehr}(P; z) := \sum_{t \geq 0} L_P(t)z^t = \frac{h^*(P; z)}{(1 - z^q)^{d+1}}, \]

where \( h^*(P; z) \) is a polynomial of degree less than \( q(d + 1) \) called the \( h^* \)-polynomial of \( P \).
Let \( P = \text{conv} \{ (-\frac{1}{2}, 1), (-\frac{1}{2}, -1), (\frac{1}{2}, 1), (\frac{1}{2}, -1) \} \).
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$L_P(t) = \begin{cases} 2t^2 + 3t + 1 & \text{when } t \text{ is even}, \\ 2t^2 + t & \text{when } t \text{ is odd}. \end{cases}$
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$\text{Ehr}(P; z) = \sum_{t \geq 0} L_P(t)z^t$

$= \sum_{t \geq 0} (2t^2 + 3t + 1)z^t + \sum_{t \geq 1} (2t^2 + t)z^t$

$= \frac{3z^4 + 12z^2 + 1}{(1 - z^2)^3} + \frac{z^5 + 12z^3 + 3z}{(1 - z^2)^3}$

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Theorem: (Ehrhart–Macdonald Reciprocity, 1971)

Let $P$ be a rational polytope. Then

$$L_P(-t) = (-1)^d L_P \circ (t).$$

Similarly, $Ehr(P; 1z) = (-1)^{d+1} Ehr(P \circ z)$.

Theorem: (Stanley's Non-negativity Result, 1980)

For a rational $d$-polytope with $Ehr(P; z) = h^\ast(P; z)(1 - zq)^d$, the coefficients of the $h^\ast$-polynomial are non-negative integers, i.e., $h^\ast_j \geq 0$.

Theorem: (Stanley's Monotonicity Result, 1993)

For $P \subseteq Q$, where $qP$ and $qQ$ are integral for some $q \in \mathbb{Z} > 0$, $h^\ast(P) \leq h^\ast(Q)$. 
Theorem: (Ehrhart–Macdonald Reciprocity, 1971)

Let $P$ be a rational polytope. Then $L_P(-t) = (-1)^d L_{P^o}(t)$. Similarly, $\text{Ehr}(P; \frac{1}{z}) = (-1)^{d+1} \text{Ehr}(P^o; z)$. 
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**Theorem:** (Stanley’s Non-negativity Result, 1980)
For a rational $d$-polytope with $\text{Ehr}(P; z) = \frac{h^*(P; z)}{(1-z^q)^{d+1}}$, the coefficients of the $h^*$-polynomial are non-negative integers, i.e., $h_j^* \geq \mathbb{Z}_{\geq 0}$.
Ehrhart Theory of Rational Polytopes

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Theorem: (Stanley’s Monotonicity Result, 1993) For $P \subseteq Q$, where $qP$ and $qQ$ are integral for some $q \in \mathbb{Z}_{>0}$, $h^*(P) \leq h^*(Q)$. 
Goals

1. Present a generalization of a decomposition of the $h^*$-polynomial for lattice polytopes due to Betke and McMullen (1985).
   (i) Use this decomposition to provide another proof of Stanley’s Monotonicity Result.

2. Present a generalization of the $h^*$-polynomial for lattice polytopes due to Stapledon (2009).
   (i) Application of this decomposition.
A rational pointed simplicial cone is a set of the form

\[ K(W) = \left\{ \sum_{i=1}^{n} \lambda_i w_i : \lambda_i \geq 0 \right\}, \]

where \( W := \{w_1, \ldots, w_n\} \) is a set of linearly independent vectors in \( \mathbb{Z}^d \).
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Define the open parallelepiped associated with $K(W)$ as

$$\text{Box}(W) := \left\{ \sum_{i=1}^{n} \lambda_i w_i : 0 < \lambda_i < 1 \right\}.$$
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Let \( u : \mathbb{R}^d \to \mathbb{R} \) denote the projection onto the last coordinate. We then define the box polynomial as
\[ B(W; z) := \sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{u(v)}. \]
Example:

Let $W = \{(1,3), (2,3)\}$. Then $\text{Box}(W) = \{\lambda_1 (1,3) + \lambda_2 (2,3) : 0 < \lambda_1, \lambda_2 < 1\}$.

Thus, $\text{Box}(W) \cap \mathbb{Z}_2 = \{(1,2), (2,4)\}$ and its associated box polynomial is $B(W; z) = z^2 + z^4$. 
Example: Let $W = \{(1,3), (2,3)\}$. Then

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Set-Up and Notation

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$$B(\mathbf{W}; z) = z^2 + z^4.$$
Set-Up and Notation

- Define the \textit{fundamental parallelepiped} $\Pi(\mathbf{W})$ to be the half-open variant of $\text{Box}(\mathbf{W})$, namely

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For a rational polytope $P \subset \mathbb{R}^d$ with vertices $v_1, \ldots, v_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate. Then the cone of $P$ is

$$\text{cone}(P) = \left\{ \sum_{i=1}^{n} \lambda_i (v_i, 1) : \lambda_i \geq 0 \right\}.$$
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Set-Up and Notation

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- If all vertices of $T$ are rational points, define the *denominator* of $T$ to be the least common multiple of all vertex coordinate denominators of the faces of $T$. 

![Diagram of a triangulation]

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Set-Up and Notation

For each $\Delta \in T$, we define the $h$-polynomial of $\Delta$ with respect to $T$ as

$$h_T(\Delta; z) := (1 - z)^{-\dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} (z^1 - z)^{\dim(\Phi) - \dim(\Delta)},$$

where the sum is over all simplices $\Phi \in T$ containing $\Delta$.

For a simplex $\Delta$ with denominator $p$, let $W$ be the set of integral ray generators of cone($\Delta$) at height $p$. We define the $h^*$-polynomial of $\Delta$ as the generating function of the last coordinate of integer points in $\Pi(W) := \Pi(\Delta)$, that is,

$$h^*(\Delta; z) = \sum_{v \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{u(v)}.$$
For each $\Delta \in T$, we define the $h$-polynomial of $\Delta$ with respect to $T$ as

$$h_T(\Delta; z) := (1 - z)^{d - \dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left( \frac{z}{1 - z} \right)^{\dim(\Phi) - \dim(\Delta)},$$

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$$h^*(\Delta; z) = \sum_{v \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{u(v)}.$$
Let $P$ be a rational $d$-polytope and $T$ be a triangulation with denominator $q$. For an $m$-simplex $\Delta \in T$, let $W = \{(r_1, q), \ldots, (r_{m+1}, q)\}$, where the $(r_i, q)$ are the integral ray generators for cone($\Delta$) at height $q$.

Set $B(W; z) := B(\Delta; z)$ and Box($W$) := Box($\Delta$).

Lemma: Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$ and let $\Delta \in T$. Then $h^*(\Delta; z) = \sum_{\Omega \subseteq \Delta} B(\Omega; z)$.
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Set $B(W; z) =: B(\Delta; z)$ and $\text{Box}(W) =: \text{Box}(\Delta)$.

**Lemma:** Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$ and let $\Delta \in T$. Then $h^*(\Delta; z) = \sum_{\Omega \subseteq \Delta} B(\Omega; z)$. 
Theorem: (Beck–Braun–Vindas–Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$, $Ehr(P; z) = \sum_{\Omega \in T} B(\Omega; z) h(\Omega; zq)(1 - zq)^{d+1}$.

Proof Sketch:
1. Write $P$ as the disjoint union of all open nonempty simplices in $T$ ($Ehr(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} Ehr(\Delta \circ; z)$).
2. Use Ehrhart–Macdonald reciprocity.
3. Apply previous lemma.
4. Use the symmetry of box polynomials.
5. Use the definition of the $h$-polynomial.
Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

$$\text{Ehr}(P; z) = \sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q) \frac{1 - z^q}{(1 - z^q)^{d+1}}.$$
Theorem: (Beck–Braun–Vindas–Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

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Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

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- Write $P$ as the disjoint union of all open nonempty simplices in $T$ ($Ehr(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} Ehr(\Delta^\circ; z)$).
Decomposition à la Betke–McMullen

**Theorem:** (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

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**Proof Sketch:**

- Write $P$ as the disjoint union of all open nonempty simplices in $T$.
- Use Ehrhart–Macdonald reciprocity.
  
- Use $Ehr(\Delta^\circ; z)$.

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Theorem:  (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$, 

$$Ehr(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$ 

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- Use Ehrhart–Macdonald reciprocity.
- Apply previous lemma.
Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$, 

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Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

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Proof Sketch:
- Write $P$ as the disjoint union of all open nonempty simplices in $T$ ($\text{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \text{Ehr}(\Delta^\circ; z)$).
- Use Ehrhart–Macdonald reciprocity.
- Apply previous lemma.
- Use the symmetry of box polynomials.
- Use the definition of the $h$-polynomial.
**Theorem:** (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with $qP$ and $qQ$ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the $h^*$-polynomials via

$$Ehr(P; z) = \frac{h^*(P; z)}{(1 - z^q)^{\dim(P)+1}}$$

and

$$Ehr(Q; z) = \frac{h^*(Q; z)}{(1 - z^q)^{\dim(Q)+1}}.$$ 

Then $h^*_i(P; z) \leq h^*_i(Q; z)$ coefficient-wise.
Lemma: (Beck–Braun–Vindas–Méndez 2020+) Suppose $P$ is a polytope and $T$ a triangulation of $P$. Let $P \subseteq Q$ be a polytope and $T'$ be a triangulation of $Q$ such that $T'$ restricted to $P$ is $T$. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $v_1, \ldots, v_n$ of $Q$ outside the affine span of $P$ such that

1. the join $T^* \text{conv}\{v_1, \ldots, v_n\}$ is a subcomplex of $T'$ and
2. $\dim(T^* \text{conv}\{v_1, \ldots, v_n\}) = \dim(Q)$.

For every face $\Omega \in T$, the coefficient-wise inequality $h_T(\Omega; z) \leq h_{T'}(\Omega, z)$ holds.
Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose $P$ is a polytope and $T$ a triangulation of $P$. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $v_1, \ldots, v_n$ of $Q$ outside the affine span of $P$ such that

1. the join $\triangledown \text{conv}\{v_1, \ldots, v_n\}$ is a subcomplex of $T'$
2. $\dim(\triangledown \text{conv}\{v_1, \ldots, v_n\}) = \dim(Q)$.

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Rational $h^*$-Monotonicity

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Andrés R. Vindas Meléndez (U. of Kentucky)

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Rational $h^*$-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with $q_P$ and $q_Q$ integral. Then $h^*(P; z) \leq h^*(Q; z)$ coefficient-wise.

Proof Sketch: Let $P$ contained in $Q$ and let $T$ be a triangulation of $P$ and $T'$ a triangulation of $Q$ such that $T'|_P$ is $T$, where if dim($P$) < dim($Q$) the triangulation $T'$ satisfies the conditions from the previous lemma. By rational Betke–McMullen, $h^*(P; z) = \sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^{q_P})$.

Since $P \subseteq Q$, $h^*(Q; z) = \sum_{\Omega \in T'} B(\Omega; z) h_{T'|_P}(\Omega; z^{q_Q}) + \sum_{\Omega \in T' \setminus T} B(\Omega; z) h_{T'}(\Omega; z^{q_Q})$.

By the lemma, the coefficients of $\sum_{\Omega \in T} B(\Omega; z) h_{T'|_P}(\Omega; z^{q_Q})$ dominate the coefficients of $\sum_{\Omega \in T} B(\Omega; z) h_{T'}(\Omega; z^{q_Q})$. Therefore, $\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^{q_P}) \leq \sum_{\Omega \in T} B(\Omega; z) h_{T'|_P}(\Omega; z^{q_Q}) \leq \sum_{\Omega \in T} B(\Omega; z) h_{T'}(\Omega; z^{q_Q})$. 

Andrés R. Vindas Meléndez (U. of Kentucky) 12-September-2020 20 / 29
Rational $h^*$-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with $qP$ and $qQ$ integral. Then $h^*_i(P; z) \leq h^*_i(Q; z)$ coefficient-wise.

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Rational $h^*$-Monotonicity

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Rational \( h^*\)-Monotonicity

**Theorem:** (Stanley 1993) Suppose \( P \subseteq Q \) are rational polytopes with \( qP \) and \( qQ \) integral. Then \( h_i^*(P; z) \leq h_i^*(Q; z) \) coefficient-wise.

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1. Let \( P \) contained in \( Q \) and let \( T \) be a triangulation of \( P \) and \( T' \) a triangulation of \( Q \) such that \( T'|_P \) is \( T \), where if \( \dim(P) < \dim(Q) \) the triangulation \( T' \) satisfies the conditions from the previous lemma.
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- By rational Betke–McMullen, $h^*(P; z) = \sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)$. 

Andrés R. Vindas Meléndez (U. of Kentucky) 
Decompositions of $h^*(P; z)$
Rational $h^*$-Monotonicity

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Andrés R. Vindas Meléndez (U. of Kentucky)
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- $\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q) \leq \sum_{\Omega \in T} B(\Omega; z) h_{T'|_P}(\Omega; z^q) \leq \sum_{\Omega \in T} B(\Omega; z) h_{T'}(\Omega; z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega; z) h_{T'}(\Omega; z^q)$. 
Decomposition from Boundary Triangulation

Set-up:

Fix a boundary triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$. Take $\ell \in \mathbb{Z} > 0$, such that $\ell P$ contains a lattice point $a$ in its interior. Thus $(a, \ell) \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$ is a lattice point in the interior of the cone of $P$ at height $\ell$ and cone($(a, \ell))$ is the ray through the point $(a, \ell)$.

We cone over each $\Delta \in T$ and define $W = \{(r_1, q), \ldots, (r_{m+1}, q)\}$ where the $(r_i, q)$ are integral ray generators of cone($\Delta$) at height $q$.

Let $B(W; z) = B(\Delta; z)$ and $W' = W \cup \{(a, \ell)\}$ be the set of generators from $W$ together with $(a, \ell)$ and set cone($\Delta'$) to be the cone generated by $W'$, which associated box polynomial $B(W'; z) = B(\Delta'; z)$.
Set-up:

- Fix a boundary triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$. 
Decomposition from Boundary Triangulation

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- Fix a boundary triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$.
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Decomposition from Boundary Triangulation

Set-up:

- Fix a boundary triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$.
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- We cone over each $\Delta \in T$ and define $W = \{(r_1, q), \ldots, (r_{m+1}, q)\}$ where the $(r_i, q)$ are integral ray generators of cone$(\Delta)$ at height $q$. 
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- Let $B(W; z) =: B(\Delta; z)$ and $W' = W \cup \{(a, \ell)\}$ be the set of generators from $W$ together with $(a, \ell)$ and set $\text{cone}(\Delta')$ to be the cone generated by $W'$, with associated box polynomial $B(W'; z) =: B(\Delta'; z)$. 
Decomposition from Boundary Triangulation

Set-up (continued):

A point \( v \in \text{cone}(\Delta) \) can be uniquely expressed as
\[
v = \sum_{i=1}^{m+1} \lambda_i (r_i, q_i)
\]
for \( \lambda_i \geq 0 \). Define
\[
I(v) := \{ i \in [m+1] : \lambda_i \in \mathbb{Z} \}
\]
and
\[
I(v) := \mathbb{R}^{m+1} \setminus I(v).
\]

For each \( v \in \text{cone}(P) \) we associate two faces \( \Delta(v) \) and \( \Omega(v) \) of \( T \), where \( \Delta(v) \) is chosen to be the minimal face of \( T \) such that \( v \in \text{cone}(\Delta'(v)) \) and we define \( \Omega(v) := \text{conv} \{ r_i q_i : i \in I(v) \} \subseteq \Delta(v) \).
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Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Consider a rational $d$-polytope $P$ that contains an interior point $\frac{a}{\ell}$, where $a \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Then

$$h^*(P; z) = 1 - z\frac{1}{1 - z\ell} \sum_{\Omega \in T} \left( B(\Omega; z) + B(\Omega'; z) ight) h(\Omega; z\frac{1}{\ell}).$$
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$$h^*(P; z) = 1 - z^\ell \sum_{\Omega \in T} \left( B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^q) \left( 1 + z + \cdots + z^\ell - 1 \right)$$
Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Consider a rational \( d \)-polytope \( P \) that contains an interior point \( \frac{a}{\ell} \), where \( a \in \mathbb{Z}^d \) and \( \ell \in \mathbb{Z}_{>0} \). Fix a boundary triangulation \( T \) of \( P \) with denominator \( q \). Then

\[
\begin{align*}
    h^*(P; z) &= \frac{1 - z^q}{1 - z^\ell} \sum_{\Omega \in T} \left( B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^q) \\
    &= \frac{1 + z + \cdots + z^{q-1}}{1 + z + \cdots + z^{\ell-1}} \sum_{\Omega \in T} \left( B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^q).
\end{align*}
\]
Decomposition from Boundary Triangulation

\[ P = [1, 3], [2, 3] \]

Boundary triangulation with denominator 3 \((a, \ell) = (2, 4)\) simplices in \(T\): empty face \(\emptyset\) and vertices \(\Delta_1 = 1, 3\) \(\Delta_2 = 2, 3\). 

\[ W_1 = \{(1, 3)\} \quad \text{and} \quad W_2 = \{(2, 3)\} \]

For \(v \in \text{cone}(P)\) then the only options for \(\Delta(v)\) to be chosen as a minimal face of \(T\) such that \(v \in \text{cone} \Delta'\) are again to consider \(\emptyset\), \(\Delta_1\), and \(\Delta_2\). In this example, \(\Omega(v) = \Delta(v)\).

\[ \Omega \in T \quad \dim(\Omega) B(\Omega; z) B(\Omega' ; z) h(\Omega, z) \]

\[ \Delta_1 \quad 0 \quad 0 \quad 0 \quad 1 \]

\[ \Delta_2 \quad 0 \quad 0 \quad 0 \quad 1 \]

\[ \emptyset \quad -1 \quad 1 \quad z \]

\[ h^*(P; z) = 1 - z^3 \quad 1 - z^4 \quad (1 + z^3 + z^2 + z^5) = 1 + z^2 + z^4. \]
Let $P = \left[ \frac{1}{3}, \frac{2}{3} \right]$. 

Decomposition from Boundary Triangulation
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- Boundary triangulation with denominator 3
Decomposition from Boundary Triangulation

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- Boundary triangulation with denominator 3
- $(a, \ell) = (2, 4)$
Decomposition from Boundary Triangulation

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<table>
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<tr>
<th>$\Omega \in T$</th>
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Let $P = \left[ \frac{1}{3}, \frac{2}{3} \right]$.

- Boundary triangulation with denominator 3
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$$h^*(P; z) = \frac{1 - z^3}{1 - z^4} \left(1 + z^3 + z^2 + z^5\right)$$

$$= 1 + z^2 + z^4.$$
Proposition: (Beck–Braun–Vindas-Meléndez 2020+) Let $P$ be a rational $d$-polytope with denominator $q$ and Ehrhart series

$$
Ehr(P; z) = \frac{h^*(P; z)}{(1 - zq)^{d+1}}.
$$

Then $\deg h^*(P; z) = s$ if and only if $(q(d + 1) - s)P$ is the smallest integer dilate of $P$ that contains an interior lattice point.
Theorem: (Beck–Braun–Vindas Meléndez 2020+) Let $P$ be a rational $d$-polytope with denominator $q$, and let $s := \deg h^*(P; z)$. Then $h^*(P; z)$ has a unique decomposition $h^*(P; z) = a(z) + z^\ell b(z)$, where $\ell = q(d+1) - s$ and $a(z)$ and $b(z)$ are polynomials with integer coefficients satisfying $a(z) = z^{q(d+1) - 1}a(1/z)$ and $b(z) = z^{q(d+1) - 1 - \ell}b(1/z)$. Moreover, the coefficients of $a(z)$ and $b(z)$ are nonnegative.
Next, we turn our attention to the polynomial

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Theorem: (Beck–Braun–Vindas–Meléndez 2020+) Let $P$ be a rational $d$-polytope with denominator $q$, let $s := \deg h^*(P; z)$ and $\ell := q(d + 1) - s$. 

The $h^*$-vector $(h^*0, \ldots, h^*q(d + 1) - 1)$ of $P$ satisfies the following inequalities:

$\sum h^* i + \sum h^* q(d + 1) - 1 - i, i = 0, \ldots, \lfloor q(d + 1) - 1/2 \rfloor - 1,$ \hspace{1cm} (1) 

$\sum h^* s + \sum h^* s - i \geq \sum h^* i, i = 0, \ldots, q(d + 1), \hspace{1cm} (2)$
Theorem: (Beck–Braun–Vindas–Meléndez 2020+) Let $P$ be a rational $d$-polytope with denominator $q$, let $s := \deg h^*(P; z)$ and $\ell := q(d + 1) - s$. The $h^*$-vector $(h^*_0, \ldots, h^*_q(d+1)-1)$ of $P$ satisfies the following inequalities:

$$h^*_0 + \cdots + h^*_{i+1} \geq h^*_{q(d+1)-1} + \cdots + h^*_{q(d+1)-1-i}, \quad i = 0, \ldots, \left\lfloor \frac{q(d+1)-1}{2} \right\rfloor - 1,$$

$$h^*_s + \cdots + h^*_{s-i} \geq h^*_0 + \cdots + h^*_i, \quad i = 0, \ldots, q(d + 1) - 1.$$
A lattice polytope is \textit{reflexive} if its dual is also a lattice polytope.

Hibi (1992): A lattice polytope $P$ is the translate of a reflexive polytope if and only if $Ehr(P; 1_z) = (-1)^{d+1}zEhr(P; z)$ as rational functions, that is, $h^*(z)$ is palindromic.

Fiset–Kaspryzk (2008): A rational polytope $P$ whose dual is a lattice polytope has a palindromic $h^*$-polynomial.

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let $P$ be a rational polytope containing the origin. The dual of $P$ is a lattice polytope if and only if $h^*(P; z) = a(z) = b(z)$, that is, $b(z) = 0$ in the $a/b$-decomposition of $h^*(P; z)$. 
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Rational Reflexive Polytopes

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The End

¡Gracias!