Lattice polytopes from Schur and symmetric Grothendieck polynomials

Margaret Bayer
University of Kansas

Bennet Geockner
University of Washington

Su Ji Hong
University of Nebraska - Lincoln

Tyrrell McAllister
University of Wyoming

McCabe Olsen
Rose Hulman Institute of Technology

Casey Pickney
Colorado State University

Julianne Vega*
Kennesaw State University

Martha Yip
University of Kentucky

Acknowledgements: Support for the 2019 GRWC was provided by NSF-DMS grants 1604458 and 1923238, NSA grant H98230-18-1-0017, Simons Foundation Collaboration grants #426971 (to M. Ferrara) and #315347 (to J. Martin) and awards from the Combinatorics Foundation and the Elsevier Mathematical Sciences Sponsorship Fund.

Semistandard Young Tableau

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. A **Semistandard Young tableau** is a filling of an arrangement of boxes with λ_i boxes in the i-th row, such that the numbers are weakly increasing along the row and strictly increasing along the column.

Example

Consider the partition $\lambda = (2, 1, 0) \vdash 3$ and let $m = 3$. The semistandard Young tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Definition (Schur polynomial)

Let $\mathbf{x} = (x_1, \ldots, x_m)$. The **Schur polynomial** in m variables indexed by $\lambda \vdash n$ is

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}^{[m]}(\lambda)} \mathbf{x}^T,$$

where $\mathbf{x}^T = x_1^{d_1(T)} \cdots x_m^{d_m(T)}$ such that $d_i(T)$ is the number of times i appears in T.

Example

For $\lambda = (2,1,0) \vdash 3$. Let $m = 3$ and $\mathbf{x} = (x_1, x_2, x_3)$.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

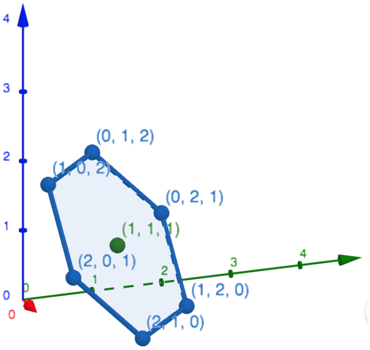
The associated Schur polynomial is

$$s_{(2,1,0)}(\mathbf{x}) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.$$

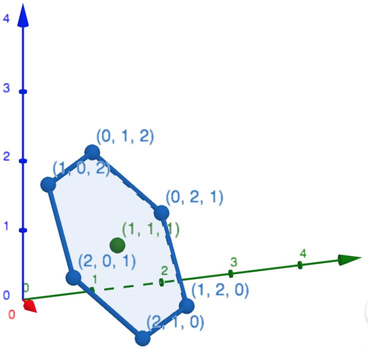
The Newton polytope $\mathrm{Newt}(s_{(2,1,0)}(\mathbf{x}))$ is the convex hull of the points $(2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,2,1), (0,1,2), (1,1,1)$.

Given a polynomial $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{C}[x_1, x_2, \ldots, x_m]$ where $\alpha \in \mathbb{Z}^m_{\geq 0}$, the **Newton polytope** $\mathrm{Newt}(f)$ conv$\{\alpha \mid c_\alpha \neq 0\}$ is the convex hull of the exponent vectors of f.

Newton Polytopes



Newton Polytopes



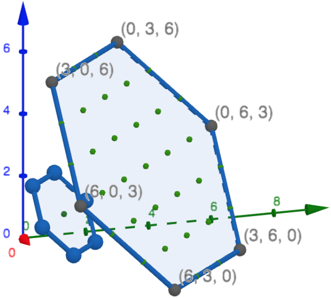
Observations:

(1) $\mathrm{Newt}(s_\lambda)$ can be realized as λ-permutohedron.

(2) The Newton polytope of a Schur polynomial is a **Saturated Newton polytope**– every lattice point $\alpha \in \mathrm{Newt}(f) \cap \mathbb{Z}^m$ appears as an exponent vector of f. [3, Monical, Tokcan, Yong].

Integer Decomposition Property

Given a positive integer t, let $t\mathcal{P}\{t\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$ be the t-th dilate of $\mathcal{P}$.

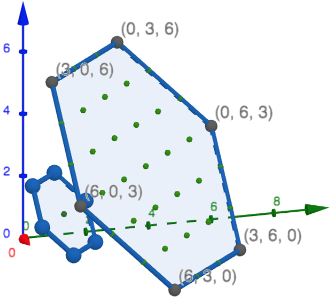


$\mathrm{Newt}(s_{(2,1,0)})$ and the 3rd dilate of $\mathrm{Newt}(s_{(2,1,0)})$.

KENNESAW STATE UNIVERSITY

Integer Decomposition Property

Given a positive integer t, let $t\mathcal{P}\{t\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$ be the t-th dilate of $\mathcal{P}$.



$\mathrm{Newt}(s_{(2,1,0)})$ and the 3rd dilate of $\mathrm{Newt}(s_{(2,1,0)})$.

Nice property: $t\mathrm{Newt}(s_\lambda) = \mathrm{Newt}(s_{t\lambda})$ for any positive integer t.

Integer Decomposition Property

Let's take the point $(2,4,3)$, which is a filling of $3\lambda = 3(2,1,0) = (6,3,0)$.

1	1	2	2	2	3
2	3	3			

Integer Decomposition Property

Let's take the point $(2, 4, 3)$, which is a filling of $3\lambda = 3(2, 1, 0) = (6, 3, 0)$.

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} + \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$(2, 4, 3) = (1, 2, 0) + (1, 1, 1) + (0, 1, 2)$$

Integer Decomposition Property

Let's take the point $(2,4,3)$, which is a filling of $3\lambda = 3(2,1,0) = (6,3,0)$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$(2,4,3) = (1,2,0) + (1,1,1) + (0,1,2)$$

Integer decomposition property (IDP): For any positive integer t and any lattice point $\mathbf{p} \in t\mathcal{P} \cap \mathbb{Z}^m$, there are t lattice points $\mathbf{v}_1, \ldots, \mathbf{v}_t \in \mathcal{P} \cap \mathbb{Z}^m$ such that $\mathbf{p} = \mathbf{v}_1 + \cdots + \mathbf{v}_t$.
Schrijver showed using generalized permutohedra and polymatroids [4].

Symmetric Grothendieck polynomials

Definition (Theorem 2.2 Lenart [2])

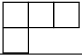
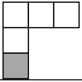
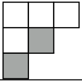
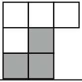
Let $\mathbf{x} = (x_1, \ldots, x_m)$ and let λ be a partition with at most m parts. The **symmetric Grothendieck polynomial** indexed by λ is

$$G_\lambda(\mathbf{x}) = \sum_{\mu \in A(\lambda)} (-1)^{|\mu/\lambda|} a_{\lambda\mu} s_\mu(\mathbf{x}).$$

where:

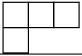
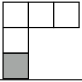
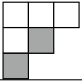
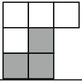
1. $\mu \supseteq \lambda$ with at most m rows,
2. the filling in the r-th row is from $\{1, \ldots, r-1\}$,
3. $a_{\lambda\mu}$ is the number of fillings of the skew shape μ/λ such that the filling increases strictly along each row and each column, and
4. $A(\lambda) = \{\mu \mid a_{\lambda\mu} \neq 0\}$.

Let $\lambda = (3,1,0) \vdash 4$, $m = 3$, and $\mathbf{x} = (x_1, x_2, x_3)$.

	$\mu = (3,1,0)$	$\emptyset$	$\mathcal{H}_4$
	$\mu = (3,2,0)$	$\boxed{1}$	$\mathcal{H}_5$
	$\mu = (3,1,1)$	$\boxed{1}$ or $\boxed{2}$	$\mathcal{H}_5$
	$\mu = (3,2,1)$	$\begin{ytableau} \none & 1 \\ 1 \end{ytableau}$ or $\begin{ytableau} \none & 1 \\ 2 \end{ytableau}$	$\mathcal{H}_6$
	$\mu = (3,2,2)$	$\begin{ytableau} \none & 1 \\ 1 & 2 \end{ytableau}$	$\mathcal{H}_7$

Symmetric Grothendieck polynomials

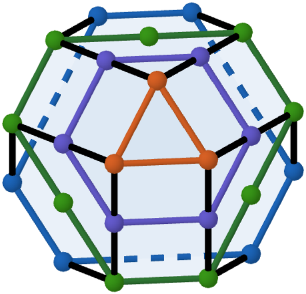
Let $\lambda = (3,1,0) \vdash 4$, $m = 3$, and $\mathbf{x} = (x_1, x_2, x_3)$.

	$\mu = (3,1,0)$	$\emptyset$	$\mathcal{H}_4$
	$\mu = (3,2,0)$	$\boxed{1}$	$\mathcal{H}_5$
	$\mu = (3,1,1)$	$\boxed{1}$ or $\boxed{2}$	$\mathcal{H}_5$
	$\mu = (3,2,1)$	$\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \end{array}$ or $\begin{array}{cc} & \boxed{1} \\ \boxed{2} & \end{array}$	$\mathcal{H}_6$
	$\mu = (3,2,2)$	$\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array}$	$\mathcal{H}_7$

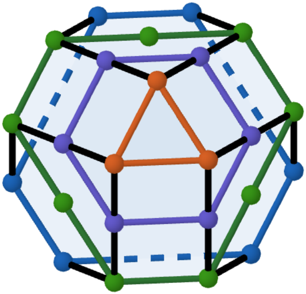
$$G_{(3,1,0)}(\mathbf{x}) =$$

$$s_{(2,1,0)}(\mathbf{x}) - \left(s_{(3,2,0)}(\mathbf{x}) + 2s_{(3,1,1)}(\mathbf{x})\right) + 2s_{(3,2,1)}(\mathbf{x}) - s_{(3,2,2)}(\mathbf{x}).$$

The Newton polytope of
$G_{(3,1,0)}(x_1, x_2, x_3) = s_{(3,1,0)} - (s_{(3,2,0)} + 2s_{(3,1,1)}) + 2s_{(3,2,1)} - s_{(3,2,2)}$.



The Newton polytope of
$G_{(3,1,0)}(x_1, x_2, x_3) = s_{(3,1,0)} - (s_{(3,2,0)} + 2s_{(3,1,1)}) + 2s_{(3,2,1)} - s_{(3,2,2)}$.



Unfortunate Property: $t\mathrm{Newt}(G_\lambda(\mathbf{x})) \neq \mathrm{Newt}(G_{t\lambda}(\mathbf{x}))$.

Unfortunate Property: $t\mathrm{Newt}(G_\lambda(\mathbf{x})) \neq \mathrm{Newt}(G_{t\lambda}(\mathbf{x}))$.

$\mathrm{Newt}(G_{(3,1,0)})$	$2\mathrm{Newt}(G_{(3,1,0)})$
$\mu = (3,1,0)$	$(6,2,0)$
$(3,2,0)$	$(6,4,0)$
$(3,1,1)$	$(6,2,2)$
$(3,2,1)$	$(6,4,2)$
$(3,2,2)$	$(6,4,4)$

Symmetric Grothendieck polynomials – Newton Polytopes

Unfortunate Property: $t\mathrm{Newt}(G_\lambda(\mathbf{x})) \neq \mathrm{Newt}(G_{t\lambda}(\mathbf{x}))$.

$\mathrm{Newt}(G_{(3,1,0)})$	$2\mathrm{Newt}(G_{(3,1,0)})$
$\mu = (3,1,0)$	$(6,2,0)$
$(3,2,0)$	$(6,4,0)$
$(3,1,1)$	$(6,2,2)$
$(3,2,1)$	$(6,4,2)$
$(3,2,2)$	$(6,4,4)$

Consider $2\lambda = (6,2,0) = \ydiagram{6,2}$. $\mathrm{Newt}(G_{(6,2,0)})$ is given by the convex hull of the union of S_3 orbit of
$\mu = (6,2,0), (6,3,0), (6,2,1), (6,3,1), (6,3,2)$.

Definition

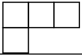
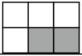
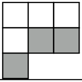
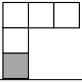
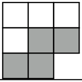
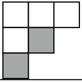
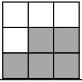
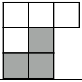
Let h be a positive integer. Let $\mathbf{x} = (x_1, \ldots, x_m)$ and let $\lambda \vdash n$ be a partition with at most m parts. The **inflated symmetric Grothendieck polynomial** indexed by λ and h is

$$G_{h,\lambda}(\mathbf{x}) = \sum_{\mu \in A(h,\lambda)} (-1)^{|\mu/\lambda|} b_{h,\lambda\mu} s_\mu(\mathbf{x}).$$

1. $\mu \supseteq \lambda$ with at most m rows,
2. the filling in the r-th row is from $\{1, \ldots, h(r-1)\}$,
3. $b_{h,\lambda\mu}$ be the number of fillings of the skew shape μ/λ such that the filling increases strictly along each row and each column, and
4. $A(h, \lambda) = \{\mu \mid b_{h,\lambda\mu} \neq 0\}$.

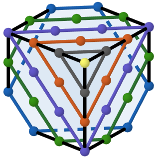
inflated Symmertic Grothendieck polynomials

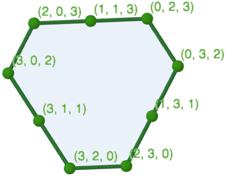
Let $h = 2$, $m = 3$, and $\lambda = (3, 1, 0)$.

	$\mu = (3,1,0)$	$\mathcal{H}_4$		$\mu = (3,3,0)$	$\mathcal{H}_6$
	$\mu = (3,2,0)$	$\mathcal{H}_5$		$\mu = (3,3,1)$	$\mathcal{H}_7$
	$\mu = (3,1,1)$	$\mathcal{H}_5$		$\mu = (3,3,2)$	$\mathcal{H}_8$
	$\mu = (3,2,1)$	$\mathcal{H}_6$		$\mu = (3,3,3)$	$\mathcal{H}_9$
	$\mu = (3,2,2)$	$\mathcal{H}_7$			

Dominating Partitions

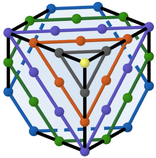
Let $h = 2$, $m = 3$, and $\lambda = (3,1,0)$. The Newton polytope of $G_{2,(3,1,0)}(x_1, x_2, x_3) =$
$s_{(3,1,0)} - 2(s_{(3,2,0)} + 4s_{(3,1,1)}) + 8s_{(3,2,1)} + 2s_{(3,3,0)} - (11s_{(3,2,2)} + 4s_{(3,3,1)})$
$+6s_{(3,3,2)} - 2s_{(3,3,3)}$.

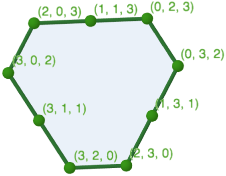




Dominating Partitions

Let $h = 2$, $m = 3$, and $\lambda = (3,1,0)$. The Newton polytope of
$G_{2,(3,1,0)}(x_1, x_2, x_3) =$
$s_{(3,1,0)} - 2(s_{(3,2,0)} + 4s_{(3,1,1)}) + 8s_{(3,2,1)} + 2s_{(3,3,0)} - (11s_{(3,2,2)} + 4s_{(3,3,1)})$
$+6s_{(3,3,2)} - 2s_{(3,3,3)}$.





For two partitions $\mu, \lambda \vdash n$ μ **dominates** λ if
$\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for every $i \geq 1$.

If $\deg G_{h,\lambda}(\mathbf{x}) = |\lambda| + N$, we say $\lambda^{(0)}, \dots, \lambda^{(N)}$ is the **sequence of dominating partitions for** $G_{h,\lambda}(\mathbf{x})$.

Integer Decomposition Property – iSGP

Let t be a positive integer. Then

$$t\mathrm{Newt}(G_{h,\lambda}(\mathbf{x})) = \mathrm{Newt}(G_{th,t\lambda}(\mathbf{x})).$$

Example (Dominating Partitions)

$\mathrm{Newt}(G_{1,(3,1,0)})$	$2\mathrm{Newt}(G_{1,(3,1,0)})$	$\mathrm{Newt}(G_{2,(6,2,0)})$
$\mu = (3,1,0)$	$(6,2,0)$	$(6,2,0)$
		$(6,3,0)$
$(3,2,0)$	$(6,4,0)$	$(6,4,0)$
		$(6,4,1)$
$(3,2,1)$	$(6,4,2)$	$(6,4,2)$
		$(6,4,3)$
$(3,2,2)$	$(6,4,4)$	$(6,4,4)$

Conclusion

Theorem

Let λ be a partition with at most m parts and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then the Newton polytope $\mathrm{Newt}(G_{h,\lambda}(\mathbf{x}))$ has the integer decomposition property.

Conclusion

Theorem

Let λ be a partition with at most m parts and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then the Newton polytope $\mathrm{Newt}(G_{h,\lambda}(\mathbf{x}))$ has the integer decomposition property.

Other Results:

- Classify which Newton polytopes of Schur and inflated Symmetric Grothendieck polynomials are reflexive.
- For the reflexive Newton polytopes of Schur polynomials we show the h^* - polynomials are unimodal.

Lattice polytopes from Schur and symmetric Grothendieck polynomials.
Arxiv: 2005.09628v2 [math.CO]

References

Laura Escobar and Alexander Yong.
Newton polytopes and symmetric Grothendieck polynomials.
C. R. Math. Acad. Sci. Paris, 355(8):831–834, 2017.

Cristian Lenart.
Combinatorial aspects of the K-theory of Grassmannians.
Ann. Comb., 4(1):67–82, 2000.

Cara Monical, Neriman Tokcan, and Alexander Yong.
Newton polytopes in algebraic combinatorics, 2017.
Preprint. *arXiv:1703.02583*.

Alexander Schrijver.
Combinatorial optimization. Polyhedra and efficiency. Vol. B, volume 24 of *Algorithms and Combinatorics*.
Springer-Verlag, Berlin, 2003.
Matroids, trees, stable sets, Chapters 39–69.

IDP– Counterexample

Not all polytopes have the integer decomposition property. For example consider the convex hull $(1,0),(0,1)$, and $(2,2)$. The second dilate contains the point $(3,3)$ but there are no two points that add to $(3,3)$.