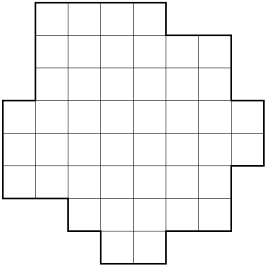
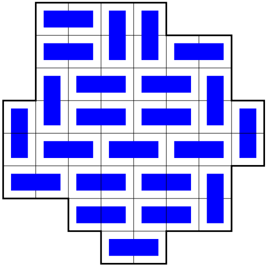
Domino tilings in dimension 3

Nicolau C. Saldanha, PUC-Rio
Includes joint work with
J. Freire, C. Klivans, P. Milet, C. Tomei and others.
Includes results due to W. Thurston and many others.

AMS Fall Central Virtual Sectional Meeting
September 13th 2020

Domino tilings in dimension 2





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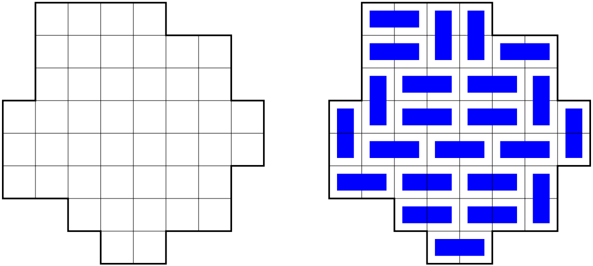

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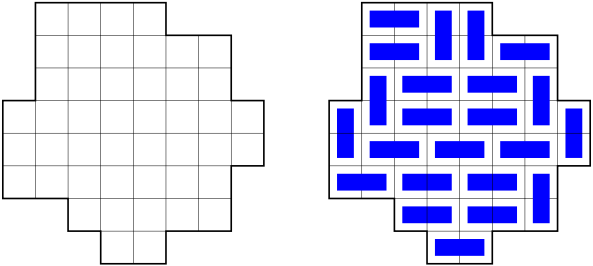

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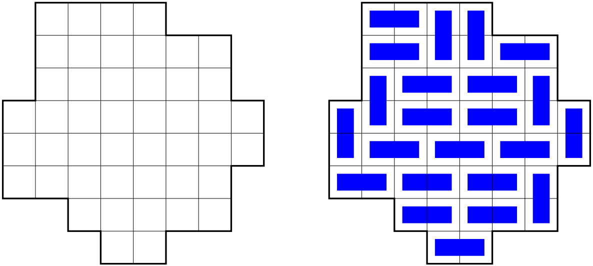

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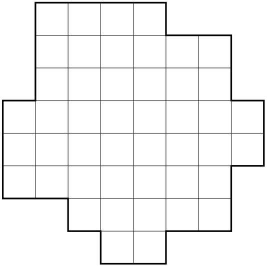

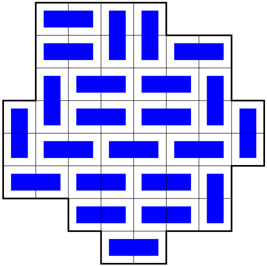

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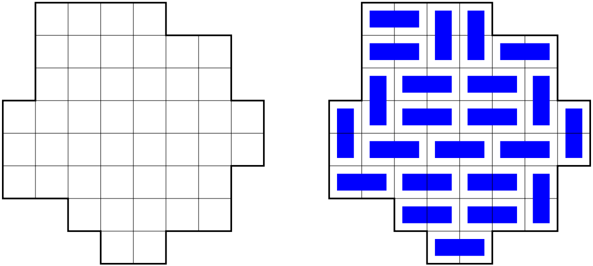

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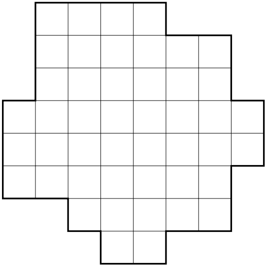

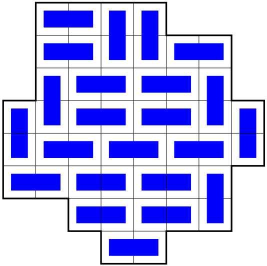

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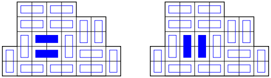
Given a quadriculated region, we want to:

- decide whether a tiling exists;
- count the possible tilings;
- classify the possible tilings;
- study the connectivity of the space of tilings by *flips*.

Flips



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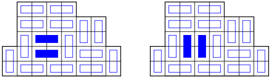

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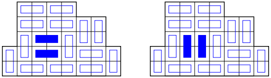

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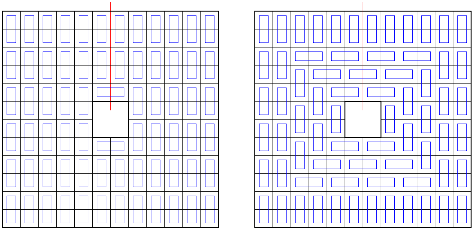
When can two tilings of the same region be joined by a finite sequence of flips? Can the two tilings below?



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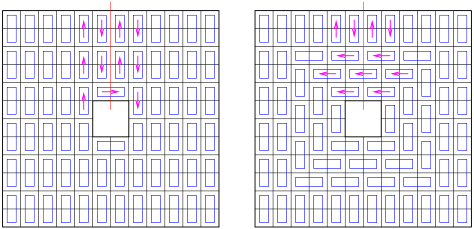

Flux is preserved by flips. The two tilings in the figure have different flux and therefore can not be joined by flips.

Flux

Theorem A (S.,Tomei,Casarin,Romualdo; extending Thurston)
Consider a planar connected quadriculated region $\mathcal{D}$.
Two domino tilings of $\mathcal{D}$ can be joined by a sequence of flips if and only if they have the same flux.

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Theorem A (S.,Tomei,Casarin,Romualdo; extending Thurston)
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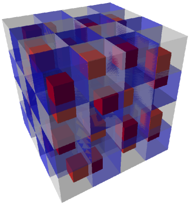
The proof is based on the concept of *height functions*.

Dominoes in dimension 3



A domino in dimension 3 is a parallelepiped with edges 2, 1 e 1, i.e., is obtained by glueing two unit cubes along a common face.

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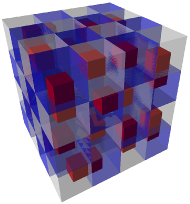

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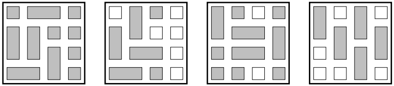
A good way to draw a tiling is by floors.
The figure shows a tiling of the box $[0,4] \times [0,4] \times [0,4]$.



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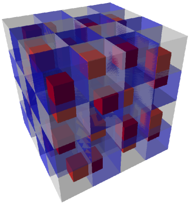

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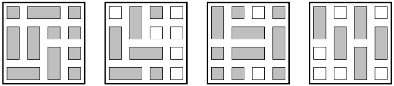

For vertical dominoes, only the half contained in the floor shown to the left is shaded.

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Even an estimate of the number a_n of tilings of a cubical box of side $2n$ is a famous open problem.

$$\lim_{n\to\infty} \frac{\log(a_n)}{n^3} = (??)$$

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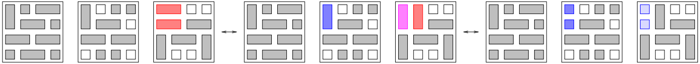

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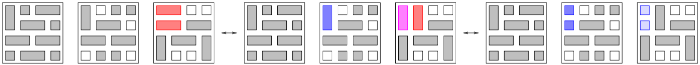

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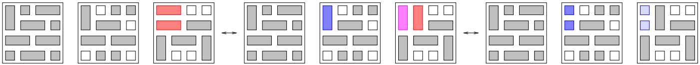

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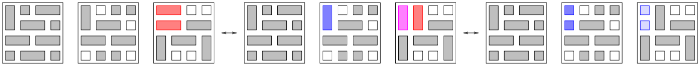

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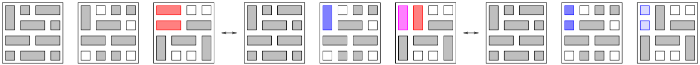
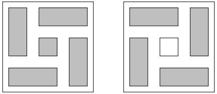
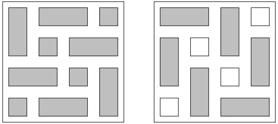


Can we prove a result similar to Theorem A? Maybe.
Is it always possible to join two tilings of a box by a finite sequence of flips? No!

There are tilings which admit no flip

It is not always possible to connect two tilings by a sequence of flips, not even if the region is a box. Some tilings admit no flip.





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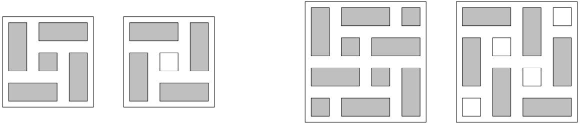
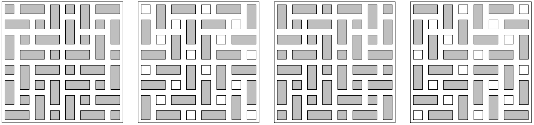


There are similar examples in larger boxes.



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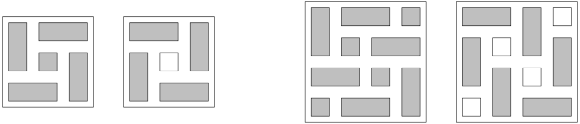

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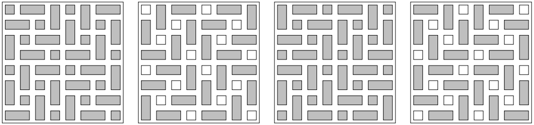
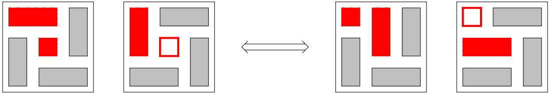


However, such examples appear to be extremely rare.

For two tilings $\mathbf{t}_0, \mathbf{t}_1$ of the same region, write $\mathbf{t}_0 \approx \mathbf{t}_1$ if there exists a sequence of flips taking $\mathbf{t}_0$ to $\mathbf{t}_1$.

Trits

A *trit* is performed by removing three dominoes, one in each possible direction, and placing then back in a different position.



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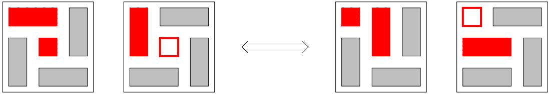

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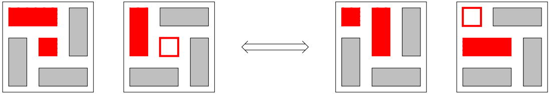

All tilings of the $4 \times 4 \times 4$ box can be joined by flips and trits.
Is this true for larger boxes?

The twist of a tiling

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Clearly, $\mathbf{t}_0 \approx \mathbf{t}_1$ implies $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$.
(Recall that $\mathbf{t}_0 \approx \mathbf{t}_1$ denotes that there exists a sequence of flips taking t_0 to t_1.
Also, flips do not alter the twist.)

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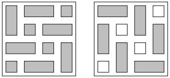
On the other hand, $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$ does not imply $\mathbf{t}_0 \approx \mathbf{t}_1$.

More about the twist

Rotations preserve twist. Reflections change the sign of the twist.



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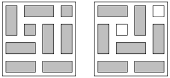

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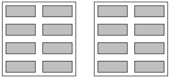

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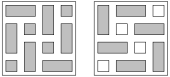

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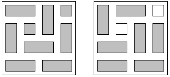

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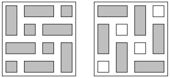

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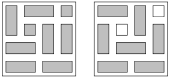

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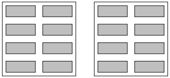

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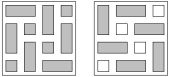

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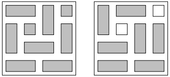

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For regions which are not contractible, the twist assumes values in $\mathbb{Z}/(d)$ where $d \in \mathbb{N} = \{0, 1, 2, \ldots\}$ is a function of the flux $(\mathbb{Z}/(0) = \mathbb{Z})$.

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That is, is it true that if two tilings have the same twist then *usually* they can be joined by a sequence of flips?

At least in certain cases, YES.

Normal distribution of the twist

For large boxes, the twist appears to follow a normal distribution.

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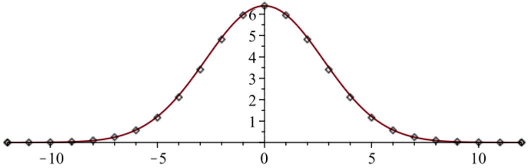

The figure shows the distribution of the twist
for the box $[0, 4] \times [0, 4] \times [0, 120]$.
The solid curve is a true gaussian, shown for comparison.
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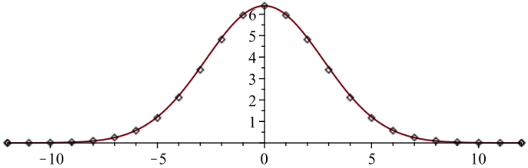

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Theorem B Let $\mathcal{D} \subset \mathbb{R}^2$ be a balanced quadriculated disk containing a 2×3 rectangle. Let $\mathbf{T}$ be a random tiling of $\mathcal{D} \times [0, N]$. When $N \to \infty$, the random variable $\mathrm{Tw}(\mathbf{T})/\sqrt{N}$ converges in distribution to a normal distribution centered at 0.

The box $4 \times 4 \times 8$

The number of tilings per twist for the box $4 \times 4 \times 8$:

$$
\begin{array}{lcl}
\mathrm{Tw} = 10 & \rightarrow & 68 \\
\mathrm{Tw} = 9 & \rightarrow & 82976 \\
\mathrm{Tw} = 8 & \rightarrow & 59065698 \\
\mathrm{Tw} = 7 & \rightarrow & 7479240824 \\
\mathrm{Tw} = 6 & \rightarrow & 789433905408 \\
\mathrm{Tw} = 5 & \rightarrow & 62605387849228 \\
\mathrm{Tw} = 4 & \rightarrow & 3436695516295322 \\
\mathrm{Tw} = 3 & \rightarrow & 115127111752195716 \\
\mathrm{Tw} = 2 & \rightarrow & 2276869405291081594 \\
\mathrm{Tw} = 1 & \rightarrow & 24306062890787668200 \\
\mathrm{Tw} = 0 & \rightarrow & 121817608970781595564 \\
\mathrm{Tw} = -1 & \rightarrow & 24306062890787668200 \\
 & \ldots &
\end{array}
$$

The box $4 \times 4 \times 12$

The number of tilings per twist for the box $4 \times 4 \times 12$:

$$
\begin{aligned}
\mathrm{Tw} = 16 &\quad\to\quad 1156\\
\mathrm{Tw} = 15 &\quad\to\quad 1718096\\
\mathrm{Tw} = 14 &\quad\to\quad 1359674808\\
&\dots\\
\mathrm{Tw} = 2 &\quad\to\quad 1778755288441774569725402318 98\\
\mathrm{Tw} = 1 &\quad\to\quad 112976714633320775075465364537 2\\
\mathrm{Tw} = 0 &\quad\to\quad 355890106778621644837267793356 1\\
&\dots
\end{aligned}
$$

Concatenation

Let $\mathbf{t}_0, \mathbf{t}_1$ be tilings of $\mathcal{D} \times [0, N_0], \mathcal{D} \times [0, N_1]$.
We denote by $\mathbf{t}_0 * \mathbf{t}_1$ the *concatenation* of these two tilings;
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For $M \in 2\mathbb{N}$, let $\mathbf{t}_{\mathrm{vert},M}$ be the vertical tiling of $\mathcal{D} \times [0, M]$.

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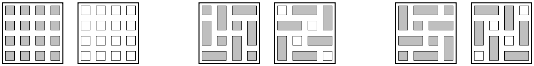
The figure shows three tilings of $\mathcal{D} \times [0, 2]$ for $\mathcal{D} = [0, 4]^2$.



The first tiling $\mathbf{t}_{\mathrm{vert},2}$ is vertical.

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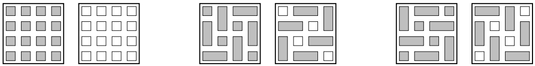

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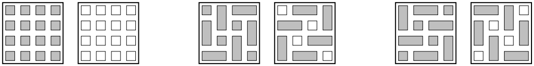

The first tiling $\mathbf{t}_{\mathrm{vert},2}$ is vertical.
The other two tilings $\mathbf{t}_0, \mathbf{t}_1$ admit no flip.
It turns out that $\mathbf{t}_0 * \mathbf{t}_{\mathrm{vert},2} \approx \mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},2}$,
i.e., there exists a finite sequence of flips joining
$\mathbf{t}_0 * \mathbf{t}_{\mathrm{vert},2}$ and $\mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},2}$.

Theorem E

(See Extra material for Theorems C, D)

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How does M depend on $\mathcal{D}$?

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Notice that $\text{Prob}[\text{Tw}(\mathbf{T}_0) = \text{Tw}(\mathbf{t}_1)]$ tends to 0 as $N \to \infty$, but not exponentially.

The $4 \times 4 \times 4$ box

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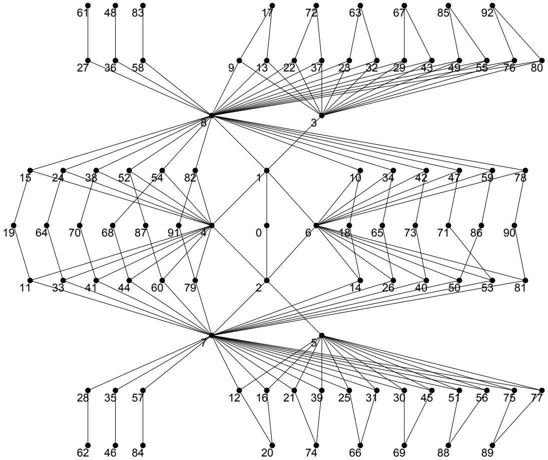
If $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$ then $\mathbf{t}_0 * \mathbf{t}_{\mathrm{vert},2} \approx \mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},2}$.
Compare with Theorem F.

The 93 connected components of the space of tilings of the $4 \times 4 \times 4$ box (via flips)



The 9 large components (5051496105 tilings)

Component	Number of tilings	Twist
0	4412646453	0
1	310185960	1
2	310185960	-1
8	8237514	2
7	8237514	-2
3	718308	2
5	718308	-2
4, 6	283044	0

The 84 small components (36000 tilings)

Component	N	Tw
27, 36, 58	2576	3
28, 35, 57	2576	-3
9, 13, 22, 23, 29, 32, 37, 43, 49, 55, 76, 80	618	3
12, 16, 21, 25, 30, 31, 39, 45, 51, 56, 75, 77	618	-3
10, 15, 24, 34, 38, 42, 47, 52, 54, 59, 78, 82	236	1
11, 14, 26, 33, 40, 41, 44, 50, 53, 60, 79, 81	236	-1
48, 61, 83	4	4
46, 62, 84	4	-4
17, 63, 67, 72, 85, 92	1	4
18, 19, 64, 65, 68, 70, 71, 73, 86, 87, 90, 91	1	0
20, 66, 69, 74, 88, 89	1	-4

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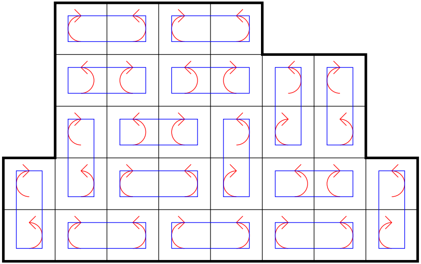
Thank you!

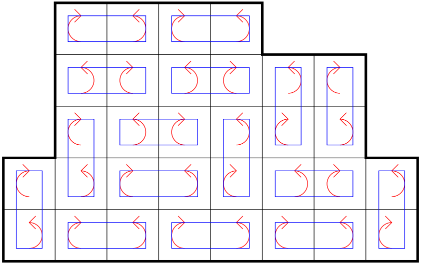
Extra material

Height functions (towards Theorem A)



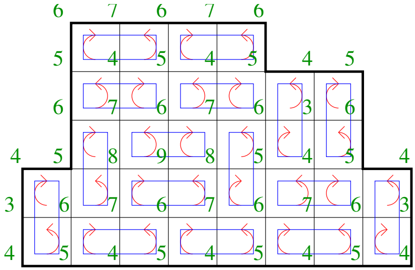
A planar domino tiling can be encoded by a *height function*.

Height functions (towards Theorem A)



A planar domino tiling can be encoded by a *height function*.
Draw clockwise and counterclockwise arrows in the squares of $\mathcal{D}$, following an alternating pattern as in the figure
(and as in a chessboard).

From a tiling to a height function



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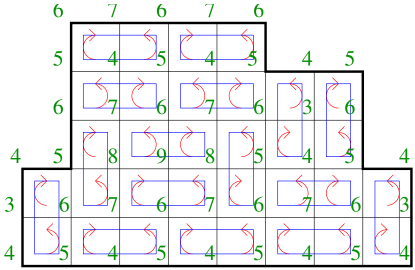

Arrows indicate whether the height function increases or decreases by 1 along edges which are not covered by dominoes.

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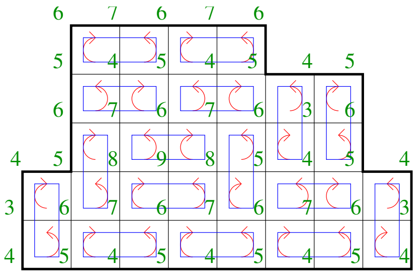
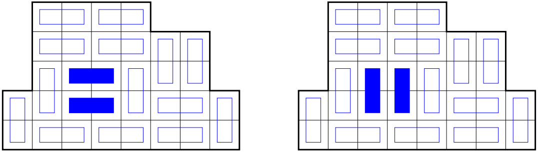
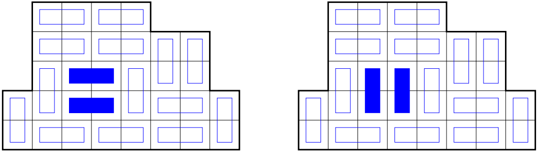


Arrows indicate whether the height function increases or decreases by 1 along edges which are not covered by dominoes. Along edges which are covered, the height function decreases or increases by 3.

Flips and height functions



Flips and height functions



Notice that when a flip is performed, the height function changes in one point only.

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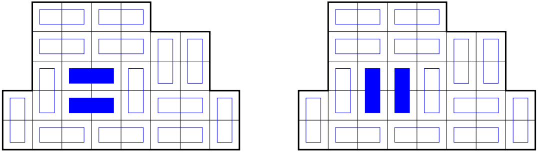

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A flip can be applied at an internal vertex if and only if that vertex is a local maximum or minimum.

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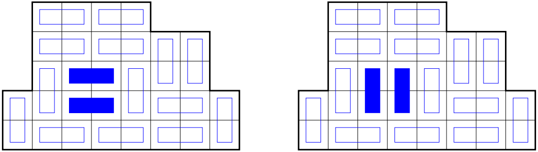
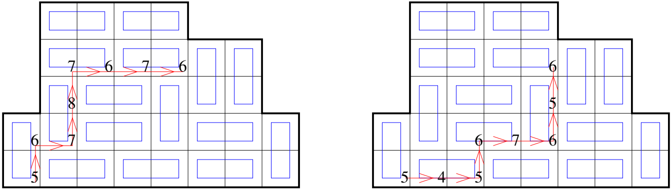


Notice that when a flip is performed, the height function changes in one point only.
A flip can be applied at an internal vertex if and only if that vertex is a local maximum or minimum.
In order to prove Theorem A, look for local maxima.

Consistency of the construction of height functions



Different paths yield the same difference.

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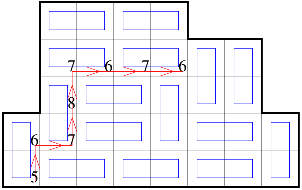
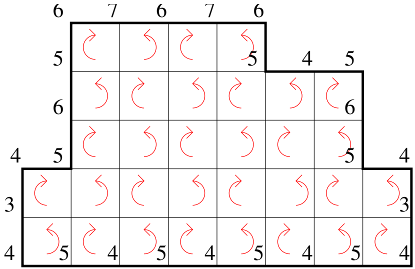


Different paths yield the same difference.
Indeed, following a closed path, the difference counts 4 times the number of black squares minus the number of white squares in the surrounded region.
(Black is counterclockwise, white is clockwise; boundaries are counterclockwise.)

The height function along the boundary of $\mathcal{D}$



The value of the height function along the boundary do not depend on the tiling.

From height function to tiling

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An integer valued function on the set of vertices of squares of $\mathcal{D}$ is a height function if and only if:

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- the function has the correct values $\mod 4$;

From height function to tiling

An integer valued function on the set of vertices of squares of $\mathcal{D}$ is a height function if and only if:

- the function has the correct values on the boundary of $\mathcal{D}$;
- the function has the correct values $\mod 4$;
- on adjacent vertices, the function differs by 1 or 3.

From height function to tiling

An integer valued function on the set of vertices of squares of $\mathcal{D}$ is a height function if and only if:

- the function has the correct values on the boundary of $\mathcal{D}$;
- the function has the correct values $\mod 4$;
- on adjacent vertices, the function differs by 1 or 3.

There exists a natural bijection between tilings and height functions.

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We thus construct the desired finite sequence of flips, proving the **Theorem**. $\square$

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Theorem C (Freire,Klivans,Milet,S.)
Consider two tilings of the same cubiculated region.
Assume they have the same flux and twist.
Then they can be joined by a finite sequence of flips
provided we are allowed to take refinements.

Short sketch of proof of Theorem C

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Generalize planar theory to quadriculated surfaces.
Deduce that the desired sequence of flips exists.

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We denote by $\mathbf{t}_0 * \mathbf{t}_1$ the *concatenation* of these two tilings;
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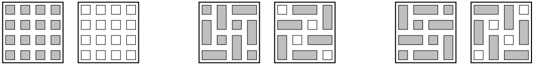
We write $\mathbf{t}_0 \sim \mathbf{t}_1$ if there exists $M \in 2\mathbb{N}$ such that $\mathbf{t}_0 * \mathbf{t}_{\mathrm{vert},M} \approx \mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},M}$.

Regular disks

The figure shows three tilings of $\mathcal{D} \times [0,2]$ for $\mathcal{D} = [0,4]^2$.



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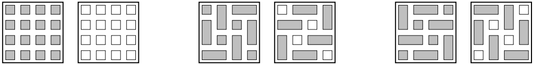

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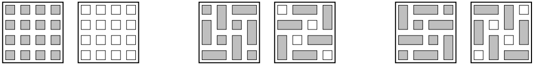

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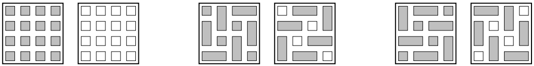

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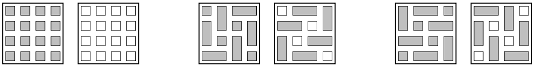

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It turns out that $\mathcal{D} = [0, 4]^2$ is regular.

Two theorems

Theorem D A rectangle $\mathcal{D} = [0, L] \times [0, M]$ (LM even) is regular if and only if $\min\{L, M\} \geq 3$.

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The complex $\mathcal{C}_{\mathcal{D}}$ (towards Theorems D, E)

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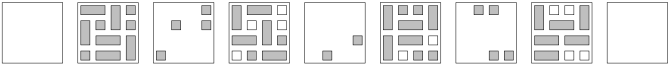
But we must be careful with edges from $\mathbf{p}_\circ$ to $\mathbf{p}_\circ$.

Tilings as paths in $\mathcal{C}_{\mathcal{D}}$

Tilings of $\mathcal{R}_N = \mathcal{D} \times [0, N]$ correspond to paths of length N in $\mathcal{C}_{\mathcal{D}}$ from $\mathbf{p}_\circ$ to $\mathbf{p}_\circ$.



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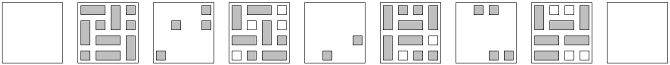

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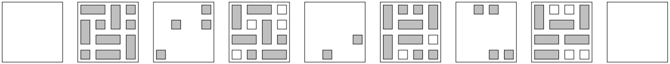

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More generally, paths of length N from p_0 to p_1 correspond to tilings of the region $\mathcal{R}_{p_0,p_1;0,N}$.

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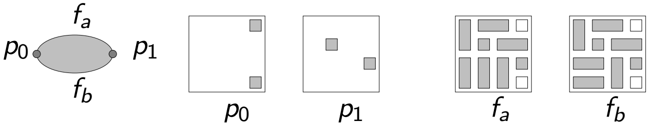

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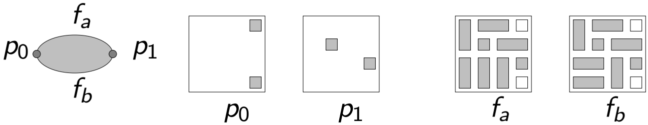

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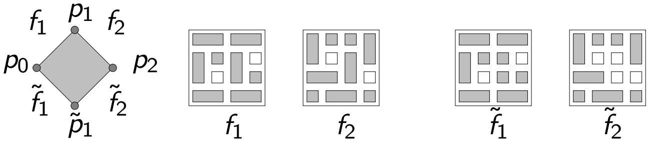

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The general case follows by induction.

The spine $\tilde{\mathcal{C}}^{\bullet}_{\mathcal{D}}$ of $\tilde{\mathcal{C}}_{\mathcal{D}}$

For a regular disk $\mathcal{D}$, there exists a simple closed curve $\Gamma : [0, N] \to \mathcal{C}_{\mathcal{D}}$ for which $\mathrm{Tw}(\Gamma)/N$ has a maximal value $c_{\mathcal{D}}$.

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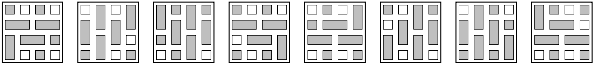

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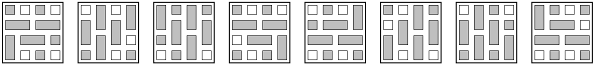

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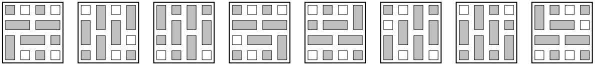

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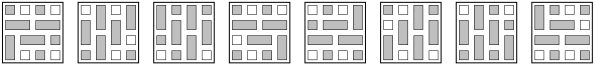

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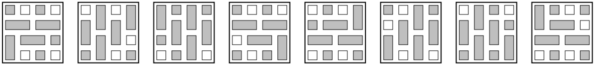

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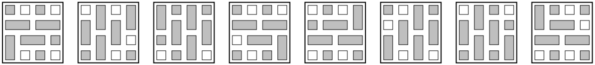

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The inclusion $\tilde{\mathcal{C}}_{\mathcal{D}}^{\bullet} \subset \tilde{\mathcal{C}}_{\mathcal{D}}$ is a quasi-isometry.

Short sketch of proof of Theorem E

Tilings $\mathbf{t}$ of $\mathcal{R}_*$ with $\mathrm{Tw}(\mathbf{t}) = t$ correspond to paths in $\tilde{\mathcal{C}}_{\mathcal{D}}$ from $\mathbf{p}_\circ$ to $\sigma^t(\mathbf{p}_\circ)$ ($\sigma : \tilde{\mathcal{C}}_{\mathcal{D}} \to \tilde{\mathcal{C}}_{\mathcal{D}}$ is a deck transformation).

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Sequences of flips joining paths correspond to homotopies with fixed endpoints.

The amount of extra vertical space needed is the length of the longest path in the homotopy.

Short sketch of proof of Theorem E (cont.)

Construct a homotopy between a base path Γ_0 and an arbitrary path Γ_1 by constructing intermediate paths $\Gamma_{\frac{s}{N+1}}$.

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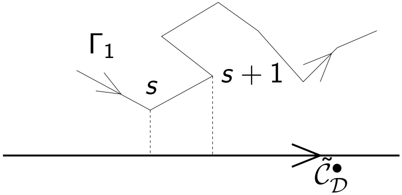

In order to define $\Gamma_{\frac{s}{N+1}}$:
Follow Γ_1 up to $\Gamma_1(s)$; then move to the spine $\tilde{\mathcal{C}}^{\bullet}_{\mathcal{D}}$ by the shortest path; then follow along the spine towards the destination.

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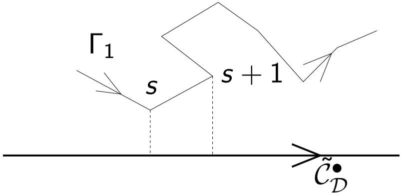

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The homotopy from $\Gamma_{\frac{s}{N+1}}$ to $\Gamma_{\frac{s+1}{N+1}}$ involves paths of bounded length, completing the proof.