Sylvan structures on near-cones

Erika Ordog
Texas A&M University
Joint work with Jack Eagon and Ezra Miller

September 13, 2020

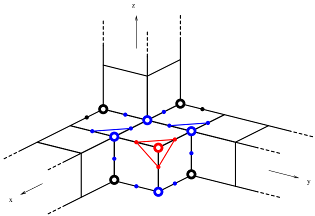
Notation and definitions

- $\mathbf{x} = x_1, x_2, \ldots, x_n$
- $S = \Bbbk[\mathbf{x}] = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \Bbbk \left\{ \mathbf{x}^{\mathbf{b}} \right\}$
- monomial: $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$
  - squarefree monomial: each b_i is either 0 or 1
- I: monomial ideal
- free module of rank r: direct sum S^r of copies of S
- a free resolution of I: a complex of free modules

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\varphi_1} F_1 \leftarrow \cdots \leftarrow F_{r-1} \xleftarrow{\varphi_r} F_r \leftarrow 0$$

  that is exact everywhere except in homological degree 0, where $I = F_0/\mathrm{im}(\varphi_1)$
- i^{th} Betti number of I in degree b: the rank $\beta_{i,\mathbf{b}}$
  $F_i = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} S(-\mathbf{b})^{\beta_{i,\mathbf{b}}}$ in a minimal free resolution of I

Koszul simplicial complexes



- $K^{\mathbf{b}} I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$
- Hochster's formula [Hochster 1977]:

$$\beta_{i,\mathbf{b}} I = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{b}} I; \Bbbk)$$

- Modules in a free resolution of I:

$$F_i = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{b}} I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{b})$$

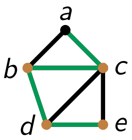
- Define a map $F_{i-1} \leftarrow F_i$ by defining a map

$$\widetilde{C}_{i-2}(K^{\mathbf{a}} I) \leftarrow \widetilde{C}_{i-1}(K^{\mathbf{b}} I)$$

that induces a well-defined homomorphism on homology

Shrubberies, stakes, and hedges

- K_i: set of i-faces of a simplicial complex K
- **shrubbery** $T_i \subseteq K_i$: set of i-faces such that ∂T_i is a $\mathbb{k}$-basis for $\widetilde{B}_{i-1}$
  - shrubbery = spanning tree or spanning forest
- **stake set** $S_{i-1} \subseteq K_{i-1}$: set of $(i-1)$-faces such that $K_{i-1} \setminus S_{i-1}$ gives a basis for $\widetilde{C}_{i-1}/\widetilde{B}_{i-1}$
- **hedge** ST_i: a pair (S_{i-1}, T_i) consisting of a stake set S_{i-1} of dimension $i-1$ and a shrubbery T_i of dimension i



$T_1 = \{ac, bc, bd, de\}$
$S_0 = \{b, c, d, e\}$
$ST_1 = (S_0, T_1)$

Splittings from shrubberies and stake sets

- A **splitting** of a complex $C_\bullet$ consists of a differential

$$d^+ = d_i^+ : C_i \to C_{i+1}$$

  such that $dd^+d = d$ and $d^+dd^+ = d^+$.

- This is equivalent to a direct sum decomposition $C_i = B'_{i-1} \oplus H_i \oplus B_i$, where B_i is the image $d(C_{i+1})$, H_i is isomorphic to $H_i(C_\bullet)$, and B'_{i-1} is isomorphic to B_{i-1}.
- Each hedge $ST_i = (S_{i-1}, T_i)$ defines a **hedge splitting** $d^+_{ST_i} : C_{i-1} \to C_i$ via
  1. $d^+d(t) = t$ for all $t \in T_i$
  2. $d^+(s) = 0$ for all $s \in \bar{S}_{i-1}$
- A **community** is a sequence of hedges $ST_\bullet = (ST_0, ST_1, ST_2, \ldots)$ such that $T_i \cap S_i = \emptyset$, and it defines a differential d^+ comprised of hedge splittings.

Minimal free resolutions from hedge splittings

Theorem (Eagon-Miller-O. 2019)

Fix a monomial ideal I. Any hedge splittings $d_{\mathbf{b}}^{+}$ of the boundary maps $d_{\mathbf{b}}$ of the Koszul simplicial complexes $K^{\mathbf{b}}I$ yield a minimal free resolution of I whose differential from homological stage $i+1$ to stage i has its component $\widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b}) \to \widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a})$ induced by the map

$$D : \widetilde{H}_i K^{\mathbf{b}} I \to \widetilde{H}_{i-1} K^{\mathbf{a}} I$$

in $\mathbb{N}^n$-degree $\mathbf{b}$ that acts on any i-cycle in $\widetilde{Z}_i K^{\mathbf{b}} I$ via

$$D \;= \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (I^{\mathbf{a}} - d_i^{\mathbf{a}+} d_i^{\mathbf{a}}) d_1^{\lambda_\ell} \Big(\prod_{j=1}^{\ell-1} d_i^{\mathbf{b}_j +} d_1^{\lambda_j} \Big) (I^{\mathbf{b}} - d_{i+1}^{\mathbf{b}} d_{i+1}^{\mathbf{b}+}),$$

where $d_1 = d_1^{e_1} + d_1^{e_2} + \ldots + d_1^{e_n}$ acts as the boundary operator, and $\lambda_j = e_k$ for some k.

Lattice paths

$$
\begin{array}{ccccccc}
\widetilde{C}_{i-1}K^{\mathbf{a}}I & \xleftarrow{d_1^{\lambda_\ell}} & & & & & \\
\partial_i^{\mathbf{a}+}\uparrow\downarrow\partial_i^{\mathbf{a}} & & \uparrow \partial_i^{\mathbf{b}_{\ell-1}+} & & & & \\
& & \widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I & \xleftarrow{d_1^{\lambda_{\ell-1}}} & & & \\
& & & \ddots & \uparrow \partial_i^{\mathbf{b}_2+} & & \\
& & & & \widetilde{C}_{i-1}K^{\mathbf{b}_2}I & \xleftarrow{d_1^{\lambda_2}} & \\
& & & & & \uparrow \partial_i^{\mathbf{b}_1+} & \partial_{i+1}^{\mathbf{b}}\downarrow\uparrow\partial_{i+1}^{\mathbf{b}+} \\
& & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_1}I \xleftarrow{d_1^{\lambda_1}} & \widetilde{C}_iK^{\mathbf{b}}I
\end{array}
$$

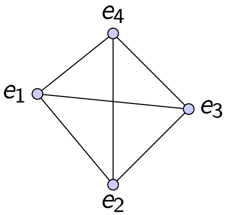
$$
D = \sum_{\lambda\in\Lambda(\mathbf{a},\mathbf{b})} (I^{\mathbf{a}} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}) d_1^{\lambda_\ell} \left(\prod_{j=1}^{\ell-1} \partial_i^{\mathbf{b}_j+} d_1^{\lambda_j} \right) (I^{\mathbf{b}} - \partial_{i+1}^{\mathbf{b}}\partial_{i+1}^{\mathbf{b}+}),
$$

Stable ideals

- For a monomial $\mathbf{x}^{\mathbf{b}}$, let $m(\mathbf{b})$ be the maximum index of a nonzero entry of $\mathbf{b}$.
- A monomial ideal I is **stable** if for every monomial $\mathbf{x}^{\mathbf{b}} \in I$, $\mathbf{x}^{\mathbf{b}-e_{m(\mathbf{b})}+e_i} \in I$ for all $1 \leq i < m(\mathbf{b})$.
  - Example: $I = \langle x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3 \rangle$
- Recall: $K^{\mathbf{b}}I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$
  - If $e_{m(\mathbf{b})} \not\leq \tau$, then $m(\mathbf{b}) = m(\tau)$ and $\mathbf{x}^{\mathbf{b}-\tau-e_{m(\mathbf{b})}+e_i} \in I$, so $\tau + e_{m(\mathbf{b})} - e_i \in K^{\mathbf{b}}I$
- If I is stable, $K^{\mathbf{b}}I$ is a **near-cone**.

Near-cones

- A simplicial complex Δ on the vertices $\{e_1, \ldots, e_n\}$ is a *near-cone* if for every $\tau \in \Delta$ such that $e_n \not\preceq \tau$, then $\tau - e_j + e_n \in \Delta$ for all $e_j \preceq \tau$. For a near-cone Δ, define $B(\Delta) = \{\tau \in \Delta \mid \tau + e_n \notin \Delta\}$.
- Example:



$B(\Delta) = \{e_1e_2, e_1e_3, e_2e_3\}$
- Proposition [Björner-Kalai 88] The faces in $B(\Delta)$ are maximal.

Hedges in near-cones

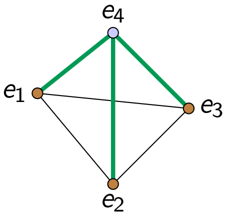
- Proposition (with Eagon and Miller): In the Koszul complex $K^{\mathbf{b}}I$, the set

$$S_{i-1} = \{\tau \mid e_{m(\mathbf{b})} \not\leq \tau, \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}}I\}$$

  for faces τ of dimension $i-1$ is a stake set of dimension $i-1$, and the set

$$T_i = \{\tau + e_{m(\mathbf{b})} \mid \tau \in S_{i-1}\}$$

  is a shrubbery of dimension i.
- Example: $S_0 = \{e_1, e_2, e_3\}$, $T_1 = \{e_1e_4, e_2e_4, e_3e_4\}$

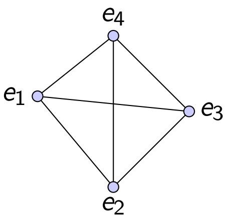


Splittings in near-cones

- Proposition (with Eagon and Miller): Let $ST_i = (S_{i-1}, T_i)$, where $S_{i-1} = \{\tau \mid e_{m(\mathbf{b})} \not\preceq \tau, \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}} I\}$ and $T_i = \{\tau + e_{m(\mathbf{b})} \mid \tau \in S_{i-1}\}$. Then

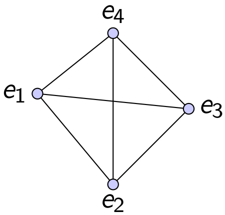
$$d^+_{ST_i}(\tau) = \begin{cases} (-1)^{|\tau|}(\tau + e_{m(\mathbf{b})}) & \text{if } m(\tau) < m(\mathbf{b}) \\ & \text{and } \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}} I \\ 0 & \text{otherwise.} \end{cases}$$

- Example: $d^+_{ST_1}(e_1) = -e_1 e_4, d^+_{ST_1}(e_4) = 0$



A canonical basis for homology

- The hedge splitting $d^+_{ST_i}$ gives a canonical basis for $\widetilde{H}_{i-1}(K^{\mathbf{b}}I; \mathbb{k})$ (computed via the projection $I - dd^+ - d^+d$).
- For each admissible symbol $e(\sigma, u)$ such that $\sigma + u = \mathbf{b}$, the basis element is $(-1)^{|\sigma|}\sigma + \sum_j c_j\tau_j$, the boundary of the face $(\sigma + e_{m(\mathbf{b})})$.
- Example: e_1e_2 corresponds to the admissible symbol $e(e_1e_2, e_3e_4)$ and the canonical basis element $e_1e_2 - e_1e_4 + e_2e_4$



Concordance with the Eliahou–Kervaire resolution

- An **admissible symbol** is of the form $e(\sigma, u)$, where u is a generator of I and $m(\sigma) < m(u)$.
- The **Eliahou–Kervaire resolution** [Eliahou-Kervaire 1990] for stable ideals has free modules F_q, which are $\mathbb{k}[\mathbf{x}]$-modules generated by the admissible symbols $e(\sigma, u)$ with $|\sigma| = q$.
- The differential $d : F_q \to F_{q-1}$ is given by
$$d(e(\sigma, u)) = \sum_{r=1}^{q} x_{i_r}(-1)^r e(\sigma_r, u) - \sum_{r \in A(\sigma; u)} (-1)^r y_r e(\sigma_r, u_r).$$
- When computing the sylvan differential, the only lattice paths that give nonzero coefficients in the image are all lattice paths of length one that move back in the direction of faces of σ and lattice paths $(\mathbf{b}_\ell, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$ where $\mathbf{b}_{i-1} - \mathbf{b}_i = m(\mathbf{b}_{i-1})$.
- Computing the maps along these lattice paths shows concordance with the Eliahou–Kervaire resolution.

References

A. Björner and G. Kalai, *An extended Euler–Poincare theorem*, Acta. Math. 161 (1998), 279-303.

John Eagon, Ezra Miller, and Erika Ordog, *Minimal resolutions of monomial ideals* (preprint), 2019. arXiv:1906.08837.

S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Alg. 129 (1990), 1-25.

Melvin Hochster, *Cohen–Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II, Lecture notes in pure and applied mathematics **26** (1977), 171–223.