Convex Union Representable complexes

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joint work with

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Families of convex sets and their nerves

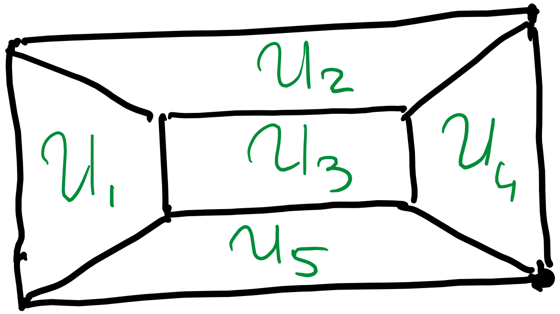
Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a collection of convex sets in $\mathbb{R}^d$

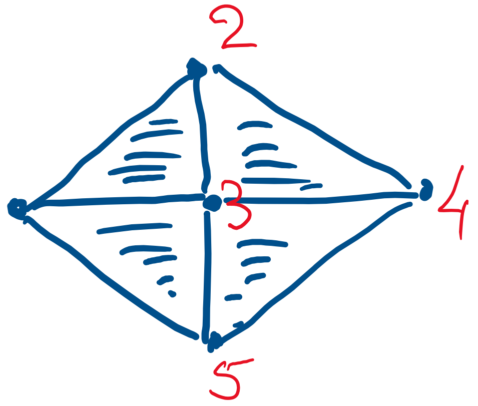
For $\sigma \subseteq [n]$, define $U_\sigma := \bigcap_{i \in \sigma} U_i$ (with $U_\emptyset = \bigcup_{i=1}^{n} U_i$)

Def The nerve of $\mathcal{U}$ is a simplicial complex

$$\mathcal{N}(\mathcal{U}) := \{\sigma \subseteq [n] : U_\sigma \neq \emptyset\}$$

Example

$\mathcal{U} =$ 

$\rightsquigarrow$ $\mathcal{N}(\mathcal{U}) =$ 

Representable and Convex Union Representable

Def A simplicial complex Δ is **d-representable** if there is a collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of convex sets in $\mathbb{R}^d$ such that $\Delta = \mathcal{N}(\mathcal{U})$

Def Δ is **d-convex union representable** (d-CUR) if there is a collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of convex open sets in $\mathbb{R}^d$ s.t. $\Delta = \mathcal{N}(\mathcal{U})$ and $U_\emptyset = U_1 \cup \ldots \cup U_n$ is convex

(**Remark** Requiring all U_i be convex and closed produces the same class)

Motivation and Background

* Rich and fascinating theory of d-representable complexes (starting from Helly, ...)

* Theory of convex neural codes

Borsuk's nerve lemma $\Longrightarrow$ every CUR complex is acyclic and even contractible

<u>Thm</u> (Chen-Frick-Shiu, 2019): CUR complexes are collapsible

Question (CFS): are all collapsible complexes CUR ?

Strengthening collapsibility

Thm (Jeffs-N) Let Δ be a d-CUR complex, let $\mathcal{U}=\{U_1,\dots,U_n\}$ be a d-convex representation of Δ, and let $C\subseteq\mathbb{R}^d$ be a convex set. Then Δ collapses onto $\mathcal{N}(\{U_i\cap C\}_{i=1}^n)$

Cor (Jeffs-N) Let Δ be a CUR complex, and $\sigma\in\Delta$ a face of Δ. Then Δ collapses onto $\mathrm{st}(\sigma)$.

(Proof: Apply the theorem to Δ, $\mathcal{U}$, and $C=U_\sigma$)

Cor (JN) If Δ is CUR, then the free faces of Δ cannot all share a common vertex

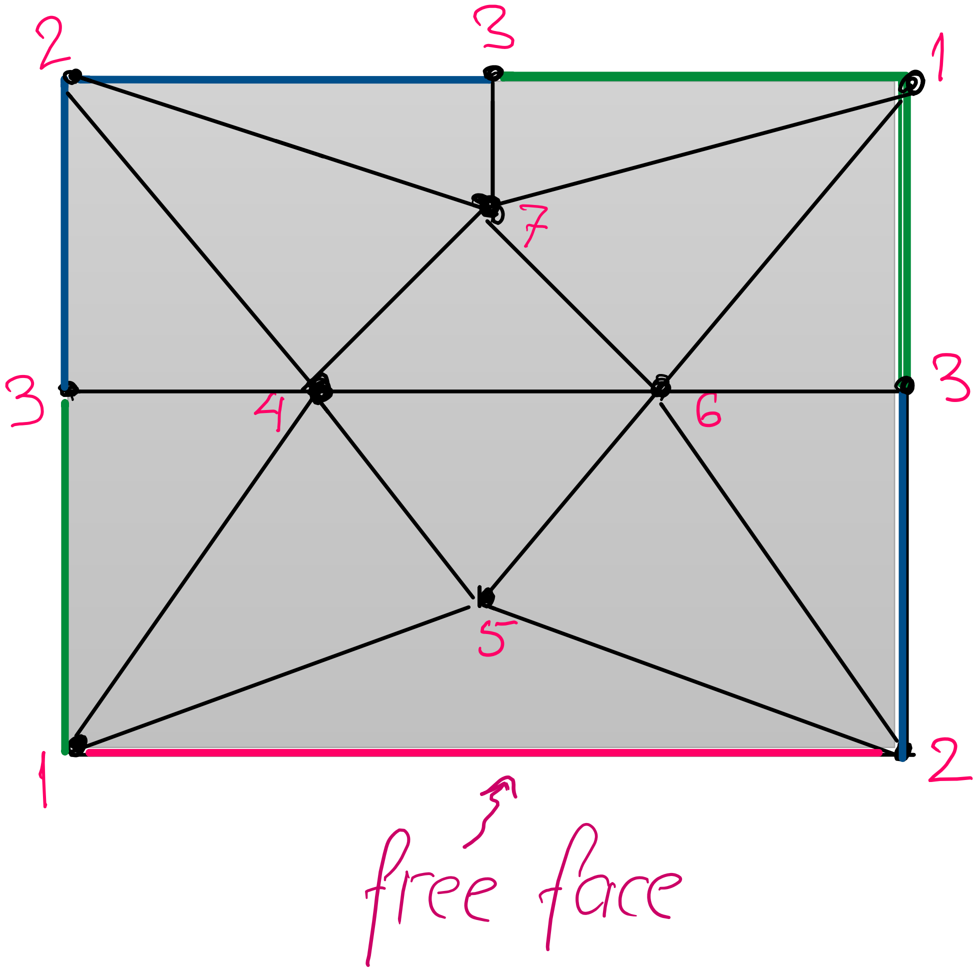
Counter-examples to Chen-Frick-Shiu's question

Thm (Adiprasito-Benedetti-Lutz, 2017)

- For all $d \geq 2$, there is a d-dim'l, pure, shellable and collapsible complex Σ_d with only <u>one</u> free face
- For all $d \geq 2$, there is a d-dim'l pure and non-evasive complex E_d with only two free faces that share a vertex

Cor (Jeffs-N)
- Σ_d and E_d are collapsible but not CUR
- Even the barycentric subdiv. of Σ_d is not CUR

Σ_2:



New question: what complexes are CUR ?

Necessary conditions: Alexander duality

- **Def** Let Δ be a simplicial complex on $[n]$. The Alexander dual of Δ is

$$\Delta^* := \{ \sigma \subseteq [n] : [n] \setminus \sigma \notin \Delta \}$$

- There exist collapsible complexes whose Alexander dual are not collapsible

- **Thm** (Jeffs-N) If Δ is CUR, then Δ^* is collapsible

Question Is Δ^* also CUR?

Necessary conditions: constructible-like behavior

Thm (Jeffs-N) Let Δ be a d-CUR complex, $\tau_1, \tau_2 \in \Delta$ s.t. $\tau_1 \cup \tau_2 \notin \Delta$.
Then there exist $\Delta_1 \subseteq \Delta \setminus \tau_1$, $\Delta_2 \subseteq \Delta \setminus \tau_2$ s.t.

- $\Delta = \Delta_1 \cup \Delta_2$
- Δ_1 and Δ_2 are d-CUR and $\Delta_1 \cap \Delta_2$ is $(d-1)$-CUR
- Δ collapses on Δ_1 and also on Δ_2, while Δ_1 and Δ_2 collapse on $\Delta_1 \cap \Delta_2$

Cor If Δ is not CUR, then neither is the suspension of Δ

Sufficient conditions

- A cone over any simplicial complex is CUR
- Joins of CUR complexes are CUR
- A 1-dimensional complex Δ is CUR $\Longleftrightarrow$ Δ is a tree
- All triangulations of 2-dimensional balls are CUR
- The antistar of any vertex v in the boundary complex of a simplicial d-polytope is $(d-1)$-CUR

Question: Is every shellable simplicial ball CUR ?

Open Problems

- Being pure and CUR $\not\Rightarrow$ being shellable (or constructible)
- Being pure, shellable, and collapsible $\not\Rightarrow$ being CUR
- Being pure and non-evasive $\not\Rightarrow$ being CUR

Question: is every CUR complex non-evasive?

Question: does every collapsible complex become CUR after sufficiently many barycentric subdivisions?

Question: does there exist Δ that is d-representable and $(d+1)$-CUR, but not d-CUR?

Thank you!