First, I’ll explain how to define chip-firing for standard representative matrices.
First, I’ll explain how to define chip-firing for standard representative matrices.

In this context, we get a generalization of Kirchhoff’s Matrix-Tree Theorem.
The Plan

- First, I’ll explain how to define chip-firing for standard representative matrices.

- In this context, we get a generalization of Kirchhoff’s Matrix-Tree Theorem.

- I’ll provide a family of combinatorially meaningful maps that are akin to bijections.
The Plan

- First, I’ll explain how to define chip-firing for standard representative matrices.

- In this context, we get a generalization of Kirchhoff’s Matrix-Tree Theorem.

- I’ll provide a family of combinatorially meaningful maps that are akin to bijections.

- This proof will use a geometric construction that gives a periodic tiling of space.
The Plan

- First, I'll explain how to define chip-firing for standard representative matrices.

- In this context, we get a generalization of Kirchhoff's Matrix-Tree Theorem.

- I'll provide a family of combinatorially meaningful maps that are akin to bijections.

- This proof will use a geometric construction that gives a periodic tiling of space.

- My goal is for the entire talk to be understandable to a general math audience.
A standard representative matrix $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \ M]$ for some integer matrix $M$. 

$$D = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$
Standard Representative Matrices

**Definition**

A *standard representative matrix* $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \ M]$ for some integer matrix $M$.

**Note**

Any *cell complex* or *orientable arithmetic matroid* satisfying a mild condition can be associated with a unique(ish) standard representative matrix.

$$D = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$
Standard Representative Matrices

Definition

A standard representative matrix $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \quad M]$ for some integer matrix $M$.

- The bases of $D$ are the linear independent sets of $r$ columns (for this talk, over $\mathbb{R}$). The set of bases is written $B(D)$.

$$D = \begin{pmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -1 & 0
\end{pmatrix}$$
A standard representative matrix $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \ M]$ for some integer matrix $M$.

The bases of $D$ are the linear independent sets of $r$ columns (for this talk, over $\mathbb{R}$). The set of bases is written $\mathcal{B}(D)$. Here,

$$\mathcal{B}(D) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$

$$D = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$
Standard Representative Matrices

**Definition**

A *standard representative matrix* $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \quad M]$ for some integer matrix $M$.

- The *bases* of $D$ are the linear independent sets of $r$ columns (for this talk, over $\mathbb{R}$). The set of bases is written $\mathcal{B}(D)$. Here,

  $$\mathcal{B}(D) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$ 

- The *multiplicity* of a basis $B$, written $m(B)$, is the magnitude of its determinant.

\[
D = \begin{pmatrix}
v_1 & v_2 & v_3 & v_4 \\
1 & 0 & 2 & 3 \\
0 & 1 & -1 & 0
\end{pmatrix}
\]
Standard Representative Matrices

Definition

A *standard representative matrix* $D$ is an $(r \times (n + r))$ matrix of the form $[I_r \ M]$ for some integer matrix $M$.

- The *bases* of $D$ are the linear independent sets of $r$ columns (for this talk, over $\mathbb{R}$). The set of bases is written $\mathcal{B}(D)$. Here,
  \[ \mathcal{B}(D) = \{ \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\} \}. \]

- The *multiplicity* of a basis $B$, written $m(B)$, is the magnitude of its determinant. Here,
  \[
  m(\{v_1, v_2\}) = 1, \quad m(\{v_1, v_3\}) = 1, \quad m(\{v_2, v_3\}) = 2, \\
  m(\{v_2, v_4\}) = 3, \text{ and } m(\{v_3, v_4\}) = 3.
  \]

\[
D = \begin{pmatrix}
  v_1 & v_2 & v_3 & v_4 \\
  1 & 0 & 2 & 3 \\
  0 & 1 & -1 & 0
\end{pmatrix}
\]
Let $D$ be a standard representative matrix $[I_r \ M]$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}$$
Let $D$ be a standard representative matrix $[I_r \ M]$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}$$

Let $\hat{D}$ be the matrix $[-M^T \ I_n]$

$$\hat{D} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{pmatrix}$$
Let $D$ be a standard representative matrix $[I_r \quad M]$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}$$

Let $\hat{D}$ be the matrix $\begin{pmatrix} -M^T \quad I_n \end{pmatrix}$

$$\hat{D} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$\hat{D}$ relates to $D$ in several ways that we will explore on the next slide.
Let's look at some properties of $D$ and $\hat{D}$.

\[
D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}
\quad
\hat{D} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{pmatrix}
\]
Let’s look at some properties of $D$ and $\hat{D}$.

\[
D = \begin{pmatrix}
1 & 0 & 2 & 3 & 7 \\
0 & 1 & -1 & 0 & -2
\end{pmatrix}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \nquartlet
Let’s look at some properties of $D$ and $\hat{D}$.

\[
D = \begin{pmatrix}
1 & 0 & 2 & 3 & 7 \\
0 & 1 & -1 & 0 & -2
\end{pmatrix} \quad \hat{D} = \begin{pmatrix}
-2 & 1 & 1 & 0 & 0 \\
-3 & 0 & 0 & 1 & 0 \\
-7 & 2 & 0 & 0 & 1
\end{pmatrix}
\]

The rows of $\hat{D}$ are all orthogonal to each row of $D$.

If we restrict $D$ to any $r$ columns and we restrict $\hat{D}$ to the remaining $n$ columns, the determinants of these submatrices are equal up to sign.
Let’s look at some properties of $D$ and $\hat{D}$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}$$

$$\hat{D} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{pmatrix}$$

- The rows of $\hat{D}$ are all orthogonal to each row of $D$.
- If we restrict $D$ to any $r$ columns and we restrict $\hat{D}$ to the remaining $n$ columns, the determinants of these submatrices are equal up to sign.
Let’s look at some properties of $D$ and $\hat{D}$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix} \quad \quad \hat{D} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{pmatrix}$$

The rows of $\hat{D}$ are all orthogonal to each row of $D$.

If we restrict $D$ to any $r$ columns and we restrict $\hat{D}$ to the remaining $n$ columns, the determinants of these submatrices are equal up to sign.

Oxley showed that $D$ and $\hat{D}$ represent dual matroids.
Let’s look at some properties of $D$ and $\hat{D}$.

$$
D = \begin{pmatrix}
1 & 0 & 2 & 3 & 7 \\
0 & 1 & -1 & 0 & -2
\end{pmatrix}
$$

$$
\hat{D} = \begin{pmatrix}
-2 & 1 & 1 & 0 & 0 \\
-3 & 0 & 0 & 1 & 0 \\
-7 & 2 & 0 & 0 & 1
\end{pmatrix}
$$

The rows of $\hat{D}$ are all orthogonal to each row of $D$.

If we restrict $D$ to any $r$ columns and we restrict $\hat{D}$ to the remaining $n$ columns, the determinants of these submatrices are equal up to sign.

Oxley showed that $D$ and $\hat{D}$ represent dual matroids.

If we put $D$ on top of $\hat{D}$, we get an invertible square matrix of the form:

$$
\mathcal{D} = \begin{bmatrix}
I_r & N \\
-N^T & I_n
\end{bmatrix}
$$
Let $D$ be a standard representative matrix and let

$$\mathcal{D} = \begin{bmatrix} D \\ \hat{D} \end{bmatrix} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}$$
Let $D$ be a standard representative matrix and let

$$D = \begin{bmatrix} \hat{D} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}$$

**Definition**

The *sandpile group* of $D$, denoted $S(D)$, is $\mathbb{Z}^{n+r}/D^T\mathbb{Z}^{n+r}$. 
Representative Matrix Sandpile Group

Let $D$ be a standard representative matrix and let

$$\mathcal{D} = \begin{bmatrix} D \\ \hat{D} \end{bmatrix} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}$$

**Definition**

The sandpile group of $D$, denoted $S(D)$, is $\mathbb{Z}^{n+r}/\mathcal{D}^T \mathbb{Z}^{n+r}$.

- This is a generalization of the classical sandpile group of a graph, but the connection is too subtle to fit into this talk.
Representative Matrix Sandpile Group

- Let $D$ be a standard representative matrix and let

$$\mathcal{D} = \begin{bmatrix} D & \hat{D} \\ \hat{D} & \end{bmatrix} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}$$

**Definition**

The sandpile group of $D$, denoted $S(D)$, is $\mathbb{Z}^{n+r}/\mathcal{D}^T\mathbb{Z}^{n+r}$.

- This is a generalization of the classical sandpile group of a graph, but the connection is too subtle to fit into this talk.
- The following theorem is closely related to Kirchhoff’s Matrix-Tree Theorem.
Let \( D \) be a standard representative matrix and let

\[
\mathcal{D} = \begin{bmatrix} D \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \mathcal{I}_r & N \\ -N^T & \mathcal{I}_n \end{bmatrix}
\]

**Definition**

The sandpile group of \( D \), denoted \( S(D) \), is \( \mathbb{Z}^{n+r} / \mathcal{D}^T \mathbb{Z}^{n+r} \).

This is a generalization of the classical sandpile group of a graph, but the connection is too subtle to fit into this talk.

The following theorem is closely related to Kirchhoff’s Matrix-Tree Theorem.

**Theorem (Duval-Klivans-Martin, 2009)**

\[
|S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2.
\]
Cellular Matrix-Tree Theorem

Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]
Cellular Matrix-Tree Theorem

Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

\[
D = \begin{pmatrix}
 v_1 & v_2 & v_3 \\
 1 & 0 & 3 \\
 0 & 1 & 2 
\end{pmatrix}
\]
Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in B(D)} m(B)^2. \]

\[ D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \]

\[ |S(D)| = m(\{v_1, v_2\})^2 + m(\{v_1, v_3\})^2 + m(\{v_2, v_3\})^2 = \]

\[ \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 + \det \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^2 + \det \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}^2 = 1^2 + 2^2 + 3^2 = 14. \]
Cellular Matrix-Tree Theorem

**Theorem (Duval-Klivans-Martin, 2009)**

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

- When \( m(B) = 1 \) for all \( B \in \mathcal{B}(D) \), we say \( D \) represents a *regular matroid*. 
Theorem (Duval-Klivans-Martin, 2009)

$$|S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2.$$  

- When $m(B) = 1$ for all $B \in \mathcal{B}(D)$, we say $D$ represents a regular matroid.

Theorem (Merino, 1999)

When $D$ represents a regular matroid, $|S(D)| = |\mathcal{B}(D)|$.  

Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

- When \( m(B) = 1 \) for all \( B \in \mathcal{B}(D) \), we say \( D \) represents a regular matroid.

Theorem (Merino, 1999)

*When \( D \) represents a regular matroid, \( |S(D)| = |\mathcal{B}(D)|. \)*

- In 2017, Backman, Baker, and Yuen defined a family of natural bijections between \( S(D) \) and \( \mathcal{B}(D) \) for the regular matroid case.
Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

- When \( m(B) = 1 \) for all \( B \in \mathcal{B}(D) \), we say \( D \) represents a **regular matroid**.

Theorem (Merino, 1999)

When \( D \) represents a regular matroid, \( |S(D)| = |\mathcal{B}(D)| \).

- In 2017, Backman, Baker, and Yuen defined a family of natural bijections between \( S(D) \) and \( \mathcal{B}(D) \) for the regular matroid case.
- Recently, I defined a family of meaningful maps \( f : S(D) \to \mathcal{B}(D) \) for any standard representative matrix \( D \) such that for every \( B \in \mathcal{B}(D) \), we have \( |f^{-1}(B)| = m(B)^2 \).
Cellular Matrix-Tree Theorem

Theorem (Duval-Klivans-Martin, 2009)

\[ |S(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

- When \( m(B) = 1 \) for all \( B \in \mathcal{B}(D) \), we say \( D \) represents a regular matroid.

Theorem (Merino, 1999)

*When \( D \) represents a regular matroid, \( |S(D)| = |\mathcal{B}(D)|. \)

- In 2017, Backman, Baker, and Yuen defined a family of natural bijections between \( S(D) \) and \( \mathcal{B}(D) \) for the regular matroid case.

- Recently, I defined a family of meaningful maps \( f : S(D) \to \mathcal{B}(D) \) for any standard representative matrix \( D \) such that for every \( B \in \mathcal{B}(D) \), we have \( |f^{-1}(B)| = m(B)^2 \). My goal of this presentation is to share these maps.
Fundamental Parallelepipeds

**Definition**

The *fundamental parallelepiped* of a square matrix $M$ with column vectors $v_1, \ldots, v_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid 0 \leq a_i \leq 1 \right\}.$$

We use the notation $\Pi_{\bullet}(M)$ to indicate the fundamental parallelepiped of $M$. 
Definition

The fundamental parallelepiped of a square matrix $M$ with column vectors $v_1, \ldots, v_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid 0 \leq a_i \leq 1 \right\}.$$

We use the notation $\Pi_{\bullet}(M)$ to indicate the fundamental parallelepiped of $M$.

- The polytope $\Pi_{\bullet}(M)$ is also the zonotope or minkowski sum of the columns vectors that make up $M$. 
**Definition**

The *fundamental parallelepiped* of a square matrix $M$ with column vectors $v_1, \ldots, v_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid 0 \leq a_i \leq 1 \right\}.$$ 

We use the notation $\Pi_\bullet(M)$ to indicate the fundamental parallelepiped of $M$.

- The polytope $\Pi_\bullet(M)$ is also the *zonotope* or *minkowski sum* of the columns vectors that make up $M$.
- In order to construct our maps, we associate each basis with the fundamental parallelepiped of a particular matrix.
Basis Parallelepips

Let $D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ which means that $D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{pmatrix}$. 

Alex McDonough (Brown University)
Basis Parallelepipeds

Let $D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ which means that $D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{pmatrix}$.

For each basis $B \in \mathcal{B}(D)$, we get a parallelepiped $P(B)$ by replacing the first $r$ or last $n$ entries of each column of $D$ by 0 based on which columns make up $B$. 
Basis Parallelepipeds

Let \( D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \) which means that \( D = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{pmatrix} \).

For each basis \( B \in B(D) \), we get a parallelepiped \( P(B) \) by replacing the first \( r \) or last \( n \) entries of each column of \( D \) by 0 based on which columns make up \( B \) (see example).

\[
\begin{align*}
P(\{v_1, v_2\}) &= \mathbf{\Pi} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
P(\{v_1, v_3\}) &= \mathbf{\Pi} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \\
P(\{v_2, v_3\}) &= \mathbf{\Pi} \cdot \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & 0 & 0 \end{pmatrix}
\end{align*}
\]
The Tile Associated with $D$

- We call $\bigcup_{B \in \mathcal{B}(D)} P(B)$ the *tile associated with* $D$, denoted $T(D)$. 
The Tile Associated with $D$

- We call $\bigcup_{B \in \mathcal{B}(D)} P(B)$ the tile associated with $D$, denoted $T(D)$.

$$T(D) = \prod \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \prod \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \cup \prod \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & 0 & 0 \end{pmatrix}$$
The Tile Associated with $D$

- We call $\bigcup_{B \in \mathcal{B}(D)} P(B)$ the tile associated with $D$, denoted $T(D)$.

$$T(D) = \Pi_{\bullet} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \bigcup \Pi_{\bullet} \left( \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \right) \bigcup \Pi_{\bullet} \left( \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & 0 & 0 \end{bmatrix} \right)$$
The Best Theorem I’ve Ever Proven

**Theorem (M. 2020)**

*The parallelepipeds that make up $T(D)$ have non overlapping interiors.*
The Best Theorem I’ve Ever Proven

Theorem (M. 2020)

The parallelepipeds that make up $T(D)$ have non overlapping interiors. Furthermore, the translates of $T(D)$ by integer linear combinations of rows of $D$ form a non-overlapping tiling of $\mathbb{R}^{r+n}$. 

Alex McDonough (Brown University)

A Higher-Dimensional Sandpile Map

Based on arxiv.org/abs/2007.09501
The simplest case is when $n = r = 1$. 

Alex McDonough (Brown University)
The Simplest Example

- The simplest case is when $n = r = 1$.
- Here, $\mathcal{D}$ is of the form
  \[ \mathcal{D} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}. \]
The Simplest Example

- The simplest case is when $n = r = 1$.
- Here, $\mathcal{D}$ is of the form
  \[ \mathcal{D} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}. \]
- For $k = 3$, $T(D)$ is shown below.
The Simplest Example

- The simplest case is when \( n = r = 1 \).
- Here, \( D \) is of the form
  \[
  D = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}.
  \]
- For \( k = 3 \), \( T(D) \) is shown below. The translates of \( T(D) \) by integer linear combinations of \((1, k)\) and \((-k, 1)\) tile the plane.
The Simplest Example

- The simplest case is when $n = r = 1$.
- Here, $D$ is of the form
  \[
  D = \begin{bmatrix}
  1 & k \\
  -k & 1
  \end{bmatrix}.
  \]
- For $k = 3$, $T(D)$ is shown below.

For our map, we can associate each point $z \in \mathbb{Z}^2$ with a basis $B$ such that $z \in P(B)$. 
The Simplest Example

- The simplest case is when \( n = r = 1 \).
- Here, \( \mathcal{D} \) is of the form
  \[
  \mathcal{D} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}.
  \]
- For \( k = 3 \), \( T(D) \) is shown below.

For our map, we can associate each point \( z \in \mathbb{Z}^2 \) with a basis \( B \) such that \( z \in \mathcal{P}(B) \).
- To do this, we nudge the points in some generic direction and see where they end up.
The Simplest Example

- The simplest case is when \( n = r = 1 \).
- Here, \( \mathcal{D} \) is of the form
  \[
  \mathcal{D} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}.
  \]
- For \( k = 3 \), \( T(D) \) is shown below.

For our map, we can associate each point \( z \in \mathbb{Z}^2 \) with a basis \( B \) such that \( z \in P(B) \).

To do this, we nudge the points in some generic direction and see where they end up.

This construction always maps \( m(B)^2 \) points into each \( P(B) \).
Conclusion

**Theorem (M. 2020)**

For any \((r \times (r + n))\) standard representative matrix \(D\), and any generic direction vector \(w \in \mathbb{R}^{r+n}\), we constructed a natural map \(f_w : S(D) \to B(D)\) such that for every \(B \in B(D)\), we have \(|f^{-1}(B)| = m(B)^2\).
Conclusion

Theorem (M. 2020)

For any \((r \times (r + n))\) standard representative matrix \(D\), and any generic direction vector \(w \in \mathbb{R}^{r+n}\), we constructed a natural map \(f_w : S(D) \rightarrow \mathcal{B}(D)\) such that for every \(B \in \mathcal{B}(D)\), we have \(|f^{-1}(B)| = m(B)^2\).

- We did this by first constructing a polytope \(P(B)\) for each basis \(B \in \mathcal{B}(D)\) and combining them to form \(T(D)\), which periodically tiles \(\mathbb{R}^{r+n}\) by translations of \(D\).
Conclusion

**Theorem (M. 2020)**

*For any \((r \times (r + n))\) standard representative matrix \(D\), and any generic direction vector \(w \in \mathbb{R}^{r+n}\), we constructed a natural map \(f_w : S(D) \to B(D)\) such that for every \(B \in B(D)\), we have \(|f^{-1}(B)| = m(B)^2\).*

- We did this by first constructing a polytope \(P(B)\) for each basis \(B \in B(D)\) and combining them to form \(T(D)\), which periodically tiles \(\mathbb{R}^{r+n}\) by translations of \(D\).

- Then, we shift each lattice point slightly in the direction of \(w\), and see which \(P(B)\) it lands in.
Conclusion

Theorem (M. 2020)

For any \((r \times (r + n))\) standard representative matrix \(D\), and any generic direction vector \(w \in \mathbb{R}^{r+n}\), we constructed a natural map \(f_w : S(D) \to B(D)\) such that for every \(B \in B(D)\), we have \(|f^{-1}(B)| = m(B)^2\).

- We did this by first constructing a polytope \(P(B)\) for each basis \(B \in B(D)\) and combining them to form \(T(D)\), which periodically tiles \(\mathbb{R}^{r+n}\) by translations of \(D\).

- Then, we shift each lattice point slightly in the direction of \(w\), and see which \(P(B)\) it lands in.

- These maps specialize to the maps given by Backman, Baker, and Yuen.
Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.
Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.
• Here are some $r = 2$ examples computed using Sage.
Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.

Here are some $r = 2$ examples computed using Sage.

$$D = \begin{bmatrix} 1 & 0 & 1 & 3 & -4 & 5 \\ 0 & 1 & 3 & 3 & 3 & -3 \end{bmatrix}$$
Because of the structure of \( \mathcal{D} \), we can also tile \( \mathbb{R}^r \) or \( \mathbb{R}^n \) instead of \( \mathbb{R}^{r+n} \).

Here are some \( r = 2 \) examples computed using Sage.

\[
D = \begin{bmatrix} 1 & 0 & 1 & 3 & -4 & 5 \\ 0 & 1 & 3 & 3 & 3 & -3 \end{bmatrix}
\]
Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.

Here are some $r = 2$ examples computed using Sage.

$$D = \begin{bmatrix}
1 & 0 & 1 & 3 & -4 & 3 & 2 \\
0 & 1 & -3 & -2 & -1 & 0 & 1
\end{bmatrix}$$
Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.

Here are some $r = 2$ examples computed using Sage.

$$D = \begin{bmatrix} 1 & 0 & 1 & 3 & -4 & 3 & 2 \\ 0 & 1 & -3 & -2 & -1 & 0 & 1 \end{bmatrix}$$
Thanks For Listening!!!