A Hopf Monoid on Set Families

Kevin Marshall
University of Kansas
kmarsh729@ku.edu

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Overview

1. Hopf Monoids and Antipodes

2. The Hopf Monoid **SetFam**

3. The Submonoid **LOI** of Lattices of Order Ideals

Punchline: **LOI** has a simple cancellation-free antipode formula!

Hopf Monoids

A *vector species* H is a collection of vector spaces $H[I]$ for all finite sets I.

- Associative **product** ("merge"):

$$\mu_{\Phi_1,\ldots,\Phi_k} : H[\Phi_1] \otimes \cdots \otimes H[\Phi_k] \to H[\Phi_1 \sqcup \cdots \sqcup \Phi_k]$$

- Coassociative **coproduct** ("break"):

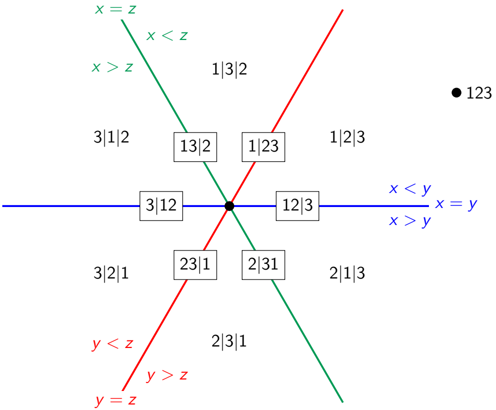
$$\Delta_{\Phi_1,\ldots,\Phi_k} : H[\Phi_1 \sqcup \cdots \sqcup \Phi_k] \to H[\Phi_1] \otimes \cdots \otimes H[\Phi_k]$$

- Compatibility: "Merging then breaking = breaking then merging"
- Antipode: Takeuchi formula

$$S(X) = \sum_{\Phi = \Phi_1 | \cdots | \Phi_k \models I} (-1)^k \mu_\Phi(\Delta_\Phi(X))$$

The Braid Arrangement

- Br_n consists of the hyperplanes $x_i = x_j$ in $\mathbb{R}^n$.
- Faces of $Br_n \iff$ *set compositions* $\Phi \models [n]$



The Aguiar–Ardila Hopf Monoid **GP**

- Aguiar and Ardila studied a Hopf monoid, **GP**, on generalized permutahedra.
- Matroids form a submonoid.
- Takeuchi formula + braid arrangement = cancellation-free antipode for **GP**
- Applications
  - Inversion of formal power series
  - Group of multiplicative characters
  - Inversion in the character group
  - Reciprocity theorems

SetFam

Grounded set family on E: collection $\mathcal{F} \subseteq 2^E$ such that $\emptyset \in \mathcal{F}$
SetFam$[I]$ = vector space spanned by grounded set families on I

Proposition

The following operations make **SetFam** *into a Hopf monoid:*

$$\mu_{A,B}(\mathcal{F}_1, \mathcal{F}_2) = \mathcal{F}_1 * \mathcal{F}_2$$
$$\Delta_{A,B}(\mathcal{F}) = \mathcal{F}|_A \otimes \mathcal{F}/_A$$

where

$$\mathcal{F}_1 * \mathcal{F}_2 = \{X \cup Y \mid X \in \mathcal{F}_1,\ Y \in \mathcal{F}_2\} \qquad (\text{“join”})$$
$$\mathcal{F}|_A = \{X \cap A \mid X \in \mathcal{F}\} \qquad (\text{“restriction”})$$
$$\mathcal{F}/_A = \{X \in \mathcal{F} \mid X \cap A = \emptyset\} \qquad (\text{“contraction”})$$

An Example: Complete Flags

A *complete flag* on $[n]$ is a set family $\{X_0, \ldots, X_n\}$ such that $X_{i-1} \subset X_i$ and $|X_i| = i$.

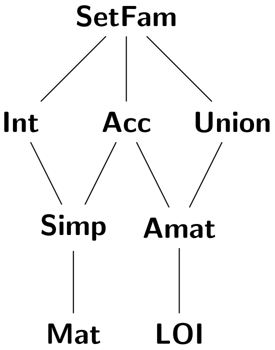
Proposition

Let $\mathcal{G} = \{\emptyset, [1], [2], \ldots, [n]\}$. *Then*

$$S(\mathcal{G}) = \sum_{B \subseteq [n]} \sum_{\substack{\Phi \models B \\ natural}} (-1)^{n-|B|+|\Phi|} \mathcal{G}_\Phi$$

where $\mathcal{G}_\Phi$ *is the family of unions of initial segments of blocks of* Φ.

Submonoids of SetFam



- **Int** $= \{\mathcal{F} : \ A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}\}$
- **Acc** = accessible set families
- **Union** $= \{\mathcal{F} : \ A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}\}$
- **Simp** = simplicial complexes
- **Amat** = **Union** $\cap$ **Acc** = antimatroids
- **Mat** = matroids
- **LOI = lattices of order ideals**

$$\textbf{LOI}[I] = \mathbb{C}\Big\langle J(P) = \{\text{order ideals of } P\} \mid P \text{ poset on } I \Big\rangle$$

Note: $J(P+Q) = J(P) * J(Q)$

A cancellation-free antipode formula

Henceforth, let P be a poset on $[n]$.
Rewrite Takeuchi's formula by grouping like terms:

$$S(J(P)) = \sum_{\Phi \models [n]} (-1)^{|\Phi|} \mu_\Phi(\Delta_\Phi(J(P)))$$
$$= \sum_{Q} J(Q) \underbrace{\left(\sum_{\Phi \in X(Q)} (-1)^{|\Phi|} \right)}_{c_Q}$$

where

$$X(Q) = \{\Phi : \ \mu_\Phi(\Delta_\Phi(J(P))) = J(Q)\}$$

1. Which posets Q arise in the sum?
2. What does c_Q mean topologically?

Terms of the antipode

Let $\Phi \models [n]$ and $a, b \in [n]$.

Say b is *betrayed* by a (w.r.t. Φ) if $a <_P b$ and $a <_\Phi b$.

$B(\Phi_i)$ = set of betrayed elements in Φ_i; $B(\Phi) = \bigcup_i B(\Phi_i)$.

Lemma

$$\mu_\Phi(\Delta_\Phi(J(P))) = \mu_\Phi\left(\bigotimes_{i=1}^{m} J(K_i)\right) = J(K_1 + \cdots + K_m)$$

where K_i is the restriction of P to $\Phi_i \setminus B(\Phi_i)$.

$X(Q)$ and $X_a(Q)$

- A *fracturing* of P is a disjoint sum of induced subposets of P. A fracturing Q is *good* if $X(Q) \neq \emptyset$.
- Suppose Q is a good fracturing of P. Let $P \setminus Q = \{b_1, \ldots, b_k\}$ and let $a = (a_1, \ldots, a_k)$ such that $a_i <_P b_i$. Define

$$X_a(Q) = \{\Phi \models [n] \mid \Phi \in X(Q) \text{ and } a_i <_\Phi b_i \ \forall i \in [k]\}.$$

- Observation:

$$X(Q) = \bigcup_a X_a(Q).$$

- Idea: Use inclusion/exclusion to calculate

$$c_Q = \sum_{\Phi \in X(Q)} (-1)^{|\Phi|}.$$

Topological properties of $X(Q)$ and $X_a(Q)$

- $X(Q)$ is an relatively-open polyhedral subfan (not necessarily convex) of the braid fan.
- $X_a(Q)$ is a convex relatively-open polyhedral fan.
- If Λ is a collection of betrayal sequences, then $\bigcap_{a \in \Lambda} X_a(Q) \neq \emptyset$.
- Replace $X(Q)$ with $X_a(Q)$ in the formula for c_Q.
- Obtain $\tilde{\chi}(\mathbb{B}^d) - \tilde{\chi}(\partial \mathbb{B}^d) = (-1)^d$.
- Apply inclusion/exclusion.

A cancellation-free antipode formula

Theorem

Suppose $J(P) \in \mathbf{LOI}$. Then a cancellation free formula for the antipode is given as a sum over good fracturings of P:

$$S(J(P)) = \sum_{Q} (-1)^{u+k} J(Q)$$

where u is the number of disjoint summands of Q and $k = |P \setminus Q|$.

- P is a chain: complete flags
- P is an antichain: $S(2^{[n]}) = (-1)^n 2^{[n]}$

Further Directions

- Use the antipode formula to study characters, polynomial invariants, etc. on **LOI**
- Cancellation-free antipode formula for other submonoids
  - Antimatroids?
  - Matroids?

Thank you!