Type polytopes and products of simplices

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Motivation

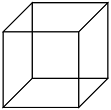
If two **polytopes** are **combinatorially isomorphic**, how “different” can they be?

Polytope: The convex hull of finitely many points (or the bounded solution set to finitely many linear inequalities)

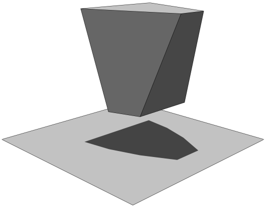
Combinatorially isomorphic: Face lattices are isomorphic

Example: Cubes

Standard cube: $C_d = [0,1]^d$

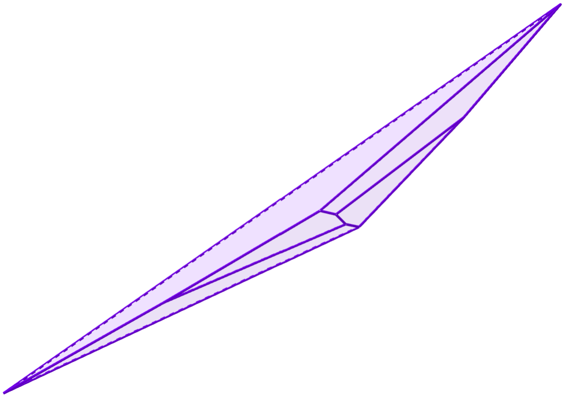


Klee–Minty cube: Simplex algorithm might have to visit all 2^d vertices



Example: Cubes

For $d \geq 3$, there exist d-cubes for which each pair of opposing facets is **perpendicular**.



A more precise question

Realization space of P: Set of all polytopes that are combinatorially isomorphic to P

- Every semialgebraic set (over $\mathbb{Z}$) is the realization space of some polytope (Mnëv, 1988).
- In 2019, Adiprasito, Kalmanovich, and Nevo showed that realization spaces of cubes are **contractible**.

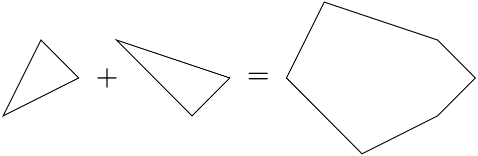
Type cone of P: Set of all polytopes that are combinatorially isomorphic to P with the same facet normal vectors

- We consider the **closure** of the original type cone (allows degeneracies).
- In 2019, Padrol, Palu, Pilaud, and Plamondon show that certain families of fans have simplicial type cones.

Minkowski sums and summands

Let $Q, R \in \mathbb{R}^d$ be polytopes. Their **Minkowski sum** is

$$Q + R = \{q + r \mid q \in Q, r \in R\}.$$



We call Q a **(weak) Minkowski summand** of P if we can find a polytope R (and a scalar λ) such that $Q + R = (\lambda)P$.

A theorem of Shephard on weak Minkowski summands

Let $V(P)$ be the vertex set of P and $E(P)$ be the edge set of P.

Theorem (Shephard)

Let $P = \{x \in \mathbb{R}^d : Ux \leq z\}$ be an irredundant inequality description for a polytope. The following are equivalent.

(i) *Q is a weak Minkowski summand of P.*

(ii) ***(Edge lengths)*** *There exists a map $\varphi : V(P) \to V(Q)$ such that for $v_i, v_j \in V(P)$ with $\{v_i, v_j\} \in E(P)$ we have $\varphi(v_i) - \varphi(v_j) = \lambda_{i,j}(v_i - v_j)$, for some $\lambda_{i,j} \in \mathbb{R}_{\geq 0}$.*

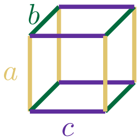
(iii) ***(Facet heights)*** *There exists an $\eta \in \mathbb{R}^m$ such that $Q = \{x \in \mathbb{R}^d : Ux \leq \eta\}$ and for any subset of rows S such that the linear system $\{\langle u_i, x\rangle = z_i, \forall i \in S\}$ defines a vertex of P, the linear system $\{\langle u_i, x\rangle = \eta_i, \forall i \in S\}$ defines a vertex in Q.*

The type cone

A **1-Minkowski weight** on P is a function $\omega : E(P) \to \mathbb{R}_{\geq 0}$ such that

$$\sum_{e \in F} \vec{e} \cdot \omega(e) = \vec{0}$$

for each two-dimensional face F of P, given any cyclic orientation of the edges of F. (The **"balancing condition."**)



The type cone

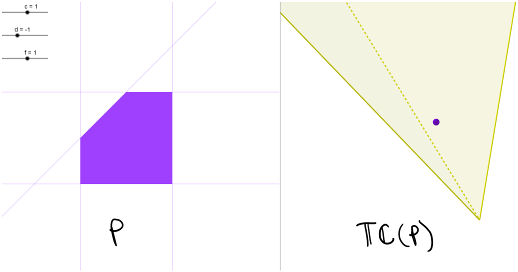
Type cone of P: $\mathbb{TC}(P) =$ Set of 1-Minkowski weights on P

Type polytope of P: $$\mathbb{TP}(P) = \left\{ \omega \in \mathbb{TC}(P) : \sum_{e \in E(P)} \omega(e) = |E(P)| \right\}$$

By Shephard's Theorem (ii), $\mathbb{TC}(P)$ parametrizes the set of weak Minkowski summands of P up to **translation**, and $\mathbb{TP}(P)$ parametrizes this set up to **translation** and **dilation**.

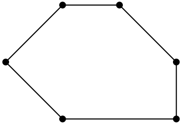
An example: Facet heights

By Shephard's Theorem (iii), we can also consider $\mathbb{TC}(P)$ and $\mathbb{TP}(P)$ in terms of facet heights.

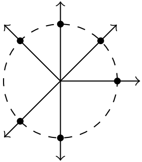


Type cones of polygons

$\mathcal{N}(P)$ = set of unit normal vectors for the facets of P



P



$\mathcal{N}(P)$

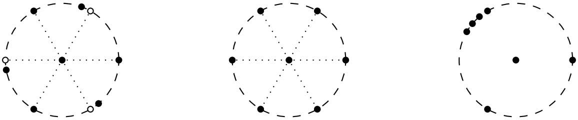
Proposition (with Castillo, Doolittle, and Ying)

For a polygon P, the faces of $\mathbb{TP}(P)$ correspond to sets $S \subseteq \mathcal{N}(P)$ such that $0 \in \operatorname{relint}(\operatorname{conv} S)$.

Corollary: Any d-polytope with $d+3$ facets is the type polytope of some polygon.

Type cones of polygons

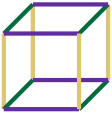
When $n > 4$, different n-gons can have non-isomorphic type polytopes. Here are three such $\mathcal{N}(P)$ for $n = 6$.

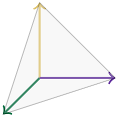


When n is even, **regular** polygons do not maximize the f-vector of the type polytope!

Type cones of cubes

Let C_d be the regular d-cube. Each **set of parallel edges** gets one parameter. Thus $\mathbb{TC}(C_d) \cong \mathbb{R}^d_{\geq 0}$ and $\mathbb{TP}(C_d)$ is a $(d-1)$-simplex.





But what about for **other cubes**?

McMullen's method

McMullen (1973) gave a way to compute $\mathbb{TP}(P)$ using intersections of convex hulls corresponding to **Gale diagrams** of the **polar dual** P°.

Theorem (McMullen)

Let P be a polytope, $\mathcal{A} = \{a_1, \cdots, a_m\}$ be the vertex set of its polar P°, and $\mathrm{Gale}(A) = \{b_1, \cdots, b_m\}$ be a Gale transform for $\mathcal{A}$. Then

$$\mathbb{TP}(P) \cong \bigcap_S \mathrm{conv}\{b_i : b_i \in S\},$$

where the intersection is over all cofacets S of $\mathcal{A}$.

This is hard to apply in general, but works well for **products of simplices**.

Our main result

Nontrivial simplex: An n-simplex for some $n > 0$.

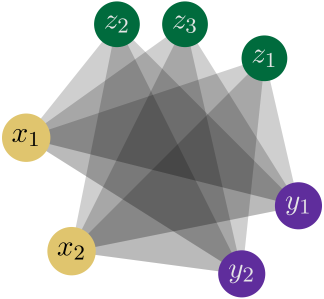
Theorem (with Castillo, Doolittle, and Ying)

If P is combinatorially isomorphic to a product of $k+1$ nontrivial simplices, $\mathbb{TP}(P)$ is a simplex of dimension k. In particular, the type polytope of any combinatorial d-cube is a $(d-1)$-simplex.

Only depends on **combinatorial type** and not **facet normals**!

"Proof"

Key step of proof: Show that the intersection of all **rainbow simplices** from a particular **rainbow configuration** is itself a simplex.



This rainbow configuration is the Gale transform of the polar of the product of nontrivial simplices. We then apply McMullen's result.

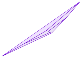
Acknowledgements

This research began at the **Graduate Research Workshop in Combinatorics 2019** at the University of Kansas.

The interactive graphics were created using **GeoGebra**. The depiction of a Klee–Minty cube appears courtesy of **Sophie Huiberts**.

The end

Thanks for listening!

$\mathbb{TP}($  $) \cong \mathbb{TP}\Big($  $\Big) \cong \mathbb{TP}\Big($  $\Big)$