Whitney Numbers for Poset Cones

AMS Central in El Paso, Texas (Online)

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September 13, 2020

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1. The Problem

2. Posets

3. Main Problem for Type A
The Problem
This presentation concerns cones $\mathcal{K}$ of arrangements of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_m\}$ in a real vector space $V \cong \mathbb{R}^n$. 

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- Each arrangement $\mathcal{A}$ dissects $V$ into connected components of the complement $V \setminus \bigcup_{i=1}^{m} H_i$ called chambers. We denote the set of chambers of $\mathcal{A}$ by $C(\mathcal{A})$. 

Hyperplane Arrangements

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- The collection $\mathcal{L}(\mathcal{A})$ of nonempty intersection subspaces $X = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$ forms a ranked poset under (reverse) inclusion. We call this the intersection poset of $\mathcal{A}$. 
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- The collection $\mathcal{L}(\mathcal{A})$ of nonempty intersection subspaces $X = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$ forms a ranked poset under (reverse) inclusion. We call this the intersection poset of $\mathcal{A}$.
- Every lower interval $[V, X] := \{Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X\}$ of $\mathcal{L}(\mathcal{A})$ forms a geometric lattice. In particular, each such $[V, X]$ is a ranked poset, with rank function given by the codimension $\text{codim}(X) := n - \dim(X)$. 
Here is an arrangement $\mathcal{A} = \{H_1, H_2, H_3\} \subseteq \mathbb{R}^2$ (left) together with the Hasse diagram of its intersection poset $\mathcal{L}(\mathcal{A})$ (right).
**Definition (Cone)**

A *cone* $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of half spaces defined by some of the hyperplanes of $\mathcal{A}$. 

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**Definition (Cone)**

A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of half spaces defined by some of the hyperplanes of $\mathcal{A}$.

**Example**

Let’s consider a cone $\mathcal{K}$ defined by $H_4$ and $H_5$ in

![Diagram showing hyperplanes $H_1$, $H_2$, $H_3$, $H_4$, and $H_5$ intersecting to form a cone]

$H_1$, $H_2$, $H_3$, $H_4$, $H_5$
Cones in an Arrangement

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![Diagram showing the intersection of half spaces defined by $H_1$, $H_3$, $H_4$, $H_5$, and $H_2$.]
Cones in an Arrangement

As with arrangements, a cone $\mathcal{K}$ in an arrangement $\mathcal{A}$ has chambers and intersections:

1. The chambers of $\mathcal{K}$ are the chambers $C(\mathcal{K}) \subseteq C(\mathcal{A})$ strictly contained in $\mathcal{K}$.

2. The nonempty intersections $L(int(\mathcal{K})) \subseteq L(int(\mathcal{A}))$ strictly contained in $\mathcal{K}$ are called interior intersections of $\mathcal{K}$.
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**Example**

![Diagram showing chambers and intersections of a cone in an arrangement](image)
Zaslavsky’s Theorem for cones

**Theorem (Zaslavsky’s Theorem for Cones)**

For a cone $K$ of an arrangement $A$ with intersection poset $L^{\text{int}}(K)$, we have

$$\# C(K) = \sum_{X \in L^{\text{int}}(K)} |\mu(V, X)| = \sum_{k=0}^{n} c_k(K)$$

where $\mu(V, X)$ denotes the Möbius function of $L^{\text{int}}(K)$ and \{ $c_k(K)$ \} are the *Whitney numbers of the cone* $K$.

In other words $\# C(K) = [\text{Poin}(K, t)]_{t=1}$, where $\text{Poin}(K, t)$ is the Poincaré polynomial of $K$, defined by

$$\text{Poin}(K, t) := \sum_{k=0}^{n} c_k(K) t^k.$$
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Here is an arrangement $\mathcal{A} = \{H_1, H_2, H_3\} \subseteq \mathbb{R}^2$ (left) together with the Hasse diagram of its intersection poset $\mathcal{L}(\mathcal{A})$ (right).
Hyperplane Arrangements

Example

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- The Poincaré polynomial of this arrangement is $\text{Poin}(\mathcal{A}, t) = t^2 + 3t + 2$. 
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- The Poincaré polynomial of this arrangement is $\text{Poin}(\mathcal{A}, t) = t^2 + 3t + 2$.
- Zaslavsky says: there are $1 + 3 + 2$ chambers.
Example

Let’s consider a cone $\mathcal{K}$ defined by $H_4$ and $H_5$ in $H_1 \cap H_2$.
Zaslavsky’s Theorem for cones

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Let’s consider a cone $\mathcal{C}$ defined by $H_4$ and $H_5$ in

$$H_4 \cap H_5 \subseteq \hat{0}$$
Zaslavsky’s Theorem for cones

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Let’s consider a cone $\mathcal{K}$ defined by $H_4$ and $H_5$ in

- $H_1$
- $H_3$
- $H_4$
- $H_5$
- $H_2$

$H_2 \cap H_3$ +1 1

$H_1$ -1 $H_2$ -1 $H_3$ -1 3

$\hat{0}$ +1 1

Zaslavsky says: there are 1 + 3 + 1 = 5 chambers in this cone.
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Let’s consider a cone $\mathcal{K}$ defined by $H_4$ and $H_5$ in

$H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \overset{0}{\rightarrow} H_2 \cap H_3 \overset{+1}{\rightarrow} 1$

$H_1 \overset{-1}{\rightarrow} H_2 \overset{-1}{\rightarrow} H_3 \overset{-1}{\rightarrow} 3$

$H_1 \overset{+1}{\rightarrow} 0 \overset{+1}{\rightarrow} 1$

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![Diagram showing the cone $\mathcal{K}$ defined by $H_4$ and $H_5$.]}

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Goal: Describe the Poincaré polynomial for cones in Type A.
Posets
The braid arrangement $A_{n-1} = \{H_{ij}\}_{1 \leq i < j \leq n}$ is the set of $\binom{n}{2}$ hyperplanes $H_{ij} = \{(x_1, \ldots, x_n) \in V = \mathbb{R}^n \mid x_i - x_j = 0\}$. 

There is an (easy) bijection between posets on $\{1, 2, \ldots, n\}$ and cones in the braid arrangement $A_{n-1}$, given by sending a poset $P$ to the cone $K_P = \{x \in V = \mathbb{R}^n \mid x_i < x_j \text{ for } i < P j\}$.

For any linear order (permutation) on $\{1, 2, \ldots, n\}$, the chamber $K_\sigma$ lies in the cone $C(K_P)$ if and only if $\sigma$ is a linear extension of $P$. 

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Cones in Type A

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- For any linear order (permutation) on $[n]$, the chamber $K_\sigma$ lies in the cone $C(K_P)$ if and only $\sigma$ is a linear extension of $P$. 
Example

Consider the cone of $A_{4-1}$ defined by a disjoint union of two chains.

The linear extensions of $P$ are:

$$\text{LinExt}(P) = \{1234, \ 1324, \ 1342, \ 3124, \ 3142, \ 3412\}$$
We can label the chambers of $\mathcal{K}_P$ by linear extensions of $P$. 

![Diagram of chambers labeled $H_{12}$, $H_{13}$, $H_{14}$, $H_{23}$, $H_{24}$, $H_{34}$]
We can label the chambers of $\mathcal{K}_P$ by linear extensions of $P$. 

Example ($A_{4-1}$)
Main Problem for Type A
Main Problem

Given a poset $P$ on $[n]$, find a statistic $\text{LinExt}(P) \xrightarrow{\text{stat}} \{0, 1, 2, \ldots\}$ interpreting

$$\#\text{LinExt}(P) = \sum_{k \geq 0} c_k(P) = [\text{Poin}(P, t)]_{t=1}$$

as follows:

$$\sum_{\sigma \in \text{LinExt}(P)} t^{\text{stat}(\sigma)} = \sum_{k \geq 0} c_k(P) t^k = \text{Poin}(P, t).$$
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Let’s motivate this with an example...
Example

Let $P$ be an antichain poset on $n$ elements. This corresponds to the full arrangement $A_{n-1}$. Then

$$1(1 + t)(1 + 2t) \cdots (1 + (n-1)t) = \sum_{\sigma \in \mathfrak{S}_n} t^{n - \# \text{cycles}(\sigma)}$$

$$= \sum_k c(n, k)t^{n-k} = \text{Poin}(P, t)$$
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We’ll generalize this example using a notion of $P$-transverse permutations.
Definition ($P$-transverse Partition)

Given a poset $P$ on $[n]$, we say that a partition $\pi$ is $P$-transverse if $\pi$ corresponds to an intersection interior to the cone $\mathcal{K}_P$. 

**Definition ($P$-transverse Permutation)**

Given a poset $P$ on $[n]$, we say that a permutation $\sigma$ is $P$-transverse if the set partition obtained by forgetting the order within the cycles is $P$-transverse.
**Definition (P-transverse Partition)**

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**Definition (P-transverse Permutation)**

Given a poset $P$ on $[n]$, we say that a permutation $\sigma$ is *P-transverse* if the set partition obtained by forgetting the order within the cycles is *P-transverse*. 
Example

Let $P$ be an antichain poset on $n$ elements. Then all permutations of $[n]$ are $P$-transverse.
Main Problem for Type A: Motivating Example

Example

Let $P$ be an antichain poset on $n$ elements. Then all permutations of $[n]$ are $P$-transverse.

Example

Let $P$ be the poset on $[4]$ with $1 < 2$ and $3 < 4$ and no other relations. Then the $P$-transverse permutations are

$(\cdot), (13), (14), (23), (24), (13)(24)$. 
Main Problem for Type A

Given a poset \( P \) on \([n]\), Zaslavsky's theorem implies that

\[
\#\text{LinExt}(P) = \#(P \text{ – transverse permutations}).
\]

Give a combinatorial bijection \( \psi \) between these two sets such that

\[
\sum_{\sigma \in \text{LinExt}(P)} t^{n - \text{cycles}(\psi(\sigma))} = \sum_{k \geq 0} c_k(P) \ t^k = \text{Poin}(P, t).
\]
Main Problem for Type A

Given a poset $P$ on $[n]$, Zaslavsky's theorem implies that

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We have such a map! Let's give an example of how it works.
Example $\psi : \text{LinExt}(P) \rightarrow \mathcal{G}^{\uparrow}(P)$

Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

$$P = \begin{array}{ccc}
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& 6 & \\
2 & & 7 \\
1 & & 3
\end{array}
\end{array}$$
Example \( \psi : \text{LinExt}(P) \to \mathcal{S}^\uparrow(P) \)

Example

Take \( \sigma = 1325476 \in \text{LinExt}(P) \) where

\[
P = \begin{array}{c}
6 & 7 \\
2 & 4 & 5 \\
1 & 3 \\
\end{array}
\]

To compute \( \psi(\sigma) \), we cut \( \sigma \) (greedily) into strings which are antichains of \( P \).
Example $\psi : \text{LinExt}(P) \to \mathcal{G}^h(P)$

Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

\[ P = \begin{array}{ccc}
1 & 3 & \text{Level 1} \\
2 & 4 & \text{Level 2} \\
6 & 7 & \text{Level 3}
\end{array} \]

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$\sigma = 1325476$. 
Example $\psi : \text{LinExt}(P) \rightarrow \mathcal{G}^H(P)$

Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

$P = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}$

Level 1

Level 2

Level 3

To compute $\psi(\sigma)$, we cut $\sigma$ (greedily) into strings which are antichains of $P$. Here

$\sigma = 1325476$.

Say that $x$ in Level $k$ is *essential* if it covers an element of Level $k - 1$. We denote the essential elements with an overline.
Example $\psi : \text{LinExt}(P) \rightarrow \mathcal{G}^h(P)$

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Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

$P = \begin{array}{ccc}
6 & 7 \\
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Level 3

Level 2

Level 1

To compute $\psi(\sigma)$, we cut $\sigma$ (greedily) into strings which are antichains of $P$. Here

$\sigma = \overline{1325476}$

Say that $x$ in Level $k$ is *essential* if it covers an element of Level $k - 1$. We denote the essential elements with an overline.
Example \( \psi : \text{LinExt}(P) \rightarrow \mathcal{G}^h(P) \)

We have \( \sigma = \overline{1325476} \).

Within each level (color block), put a left parenthesis left of each left-to-right maximum among the essential elements:

\[
\sigma = (\overline{1})(\overline{3}25(4)(76}
\]

Adding in the right parenthesis:

\[
\sigma = (1)(3)(25)(4)(76)
\]

Removing the decoration gives

\[
\psi(\sigma) = (1)(3)(25)(4)(76)
\]
Example \( \psi : \text{LinExt}(P) \to \mathcal{G}^+(P) \)

**Example**

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Within each level (color block), put a left parenthesis left of each left-to-right maximum among the essential elements:

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Adding in the right parenthesis: \( \sigma = (1)(3)(25)(4)(76) \)
Example $\psi : \text{LinExt}(P) \to S^h(P)$

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$$\sigma = (1)(3)(25)(476)$$

Adding in the right parenthesis: $\sigma = (1)(3)(25)(4)(76)$ Removing the decoration gives

$$\psi(\sigma) = (1)(3)(25)(4)(76)$$
Theorem

Given a poset $P$ on $[n]$, not only does $\psi$ give a bijection, but

$$\sum_{\sigma \in \text{LinExt}(P)} t^{n-\text{cycles}(\psi(\sigma))} = \sum_{k \geq 0} c_k(P) t^k = \text{Poin}(P, t).$$
Thanks!
References
