Cycle and circuit chip-firing on graphs

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Coauthors

- This work comes from a Mathworks project at Texas State University from summer 2019.
- Joint work with Eli Meyers, Raghav Samavedam, and Alex Yi (all high school students!)

Chip-firing on graphs

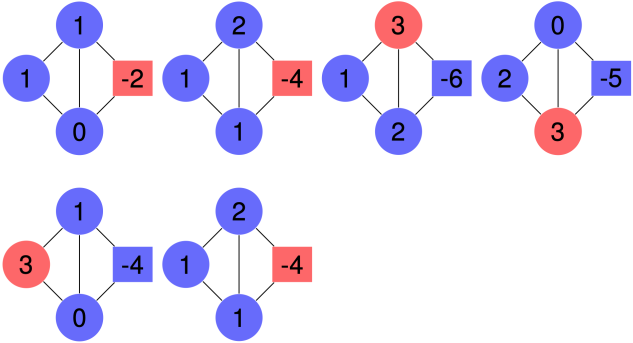
- The classical theory of *chip-firing* involves a simple game played on the vertices of a graph, with connections to
  - *self-organized crticality* in statistical physics.
  - underlying algebraic structure.
  - divisor theory in algebraic geometry.
- Recent books:
  - S. Corry and D. Perkinson, *Divisors and Sandpiles: An Introduction to Chip-Firing*, AMS, 2017.
  - C. Klivans, *The Mathematics of Chip-Firing*, CRC press, 2018.
- For us: Chip-firing on a graph G says something about $\mathcal{M}(G)^*$, the *dual* matroid of the underlying graphic matroid. What about *dual chip-firing*?

Chip-firing: The rules

Suppose G is a finite simple graph on vertex set $\{0, 1, \ldots, n\}$ with specified root vertex 0, and edge set E.

- A *configuration* of chips is a vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$.
- A vertex i can *fire* when $c_i \geq \deg(u)$.
- The vertex passes chips to each of its neighbors (one for each edge connecting it to i), resulting in a new configuration $\mathbf{c}'$.
- A configuration is *stable* if no vertex can fire.
- The root can fire only when the configuration is stable, passing a chip to each of its neighbors.

Example

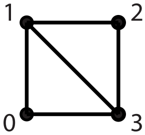


The critical group

- FACT: If G is connected all firings stabilize. Why?
- Study the dynamics - what configurations do we see 'many times'?
- Given any configuration there is a unique 'recurrent' configuration that one can obtain via stabilization.
- The set of 'critical configurations' form an abelian group $\kappa(G)$ called the 'critical group' of G.

Superstable configurations

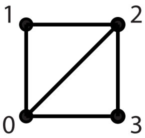
- We say $\mathbf{c}$ *superstable* if no *set* of nonroot vertices can fire simultaneously.



- In the above graph, 202 is stable but not superstable. Superstable configurations are $\{000, 100, 010, 001, 200, 002, 101, 111\}$
- Set of superstable configurations are in a simple bijection with the set of crtical configurations.
- Superstable configurations have connections to G-parking functions, Tutte polynomials (Merino), h-vectors, etc.

Some linear algebra

- The dynamics of chip-firing can be encoded in the *(reduced) Laplacian* of G, an $n \times n$ symmetric matrix that encodes G.



$$\tilde{\mathcal{L}}(G) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- Firing a vertex i corresponds to subtracting the ith row of $\tilde{\mathcal{L}}(G)$. Two configurations $\mathbf{c}$ and $\mathbf{d}$ are equivalent if $\mathbf{d} - \mathbf{c} \in \text{im}\tilde{\mathcal{L}}(G)$.
- Recovers the critical group as

$$\kappa(G) \cong \mathbb{Z}^n / \text{im}\tilde{\mathcal{L}}(G)$$

- As a corollary we see that $|\kappa(G)|$ is given by $\det \tilde{\mathcal{L}}(G) =$ the number of *spanning trees* of G.

Generalized chip-firing

Gabrielov/Dhar generalized this idea to *abelian avalanche models*.

- Consider an $n \times n$ *redistribution matrix* Δ where

$$\Delta_{ii} > 0 \text{ for all } i; \; \Delta_{ij} \leq 0 \text{ for all } i \neq j.$$

- A vector $\mathbf{h} \in \mathbb{Z}^n$ defines a configuration, and a site i can fire if $h_i \geq \Delta_{ii}$.
- In this case replace $\mathbf{h}$ with $\mathbf{h} - \Delta^T \mathbf{e}_i$.
- Notions of *stable* configurations are similar.
- Not all matrices give good firing rules - Δ is said to be *avalanche finite* if any initial configuration and firing sequence will eventually terminate in a stable configuration.

Generalized chip-firing

Guzmán and Klivans have developed a theory of chip-firing for any avalanche finite redistribution matrix L.

- Two configurations $\mathbf{f}$ and $\mathbf{g}$ are *equivalent* if $\mathbf{g} - \mathbf{f} \in \text{im} L$.
- A configuration is *z-superstable* if all entries are nonnegative and no *multiset* of sites can fire simultaneously.
- GK prove that every equivalence class contains a unique z-superstable configuration.
- In fact $\mathbf{f}$ is z-superstable if and only if $\mathbf{f}$ is the unique minimizer of

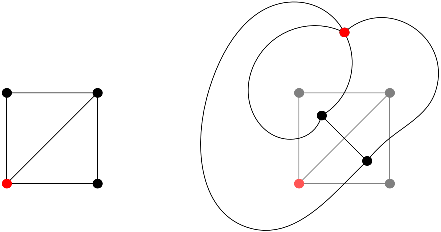
$$\min_{\mathbf{g} \sim \mathbf{f}, \mathbf{g} \geq 0} E(\mathbf{g}),$$

  where for any configuration $\mathbf{f}$ define its *energy* to be

$$E(\mathbf{f}) = \mathbf{f}^T L \mathbf{f}.$$

Back to graphs: planarity and dual chip-firing

- If G is a *planar* graph (with an embedding) what does chip-firing on the dual graph G^* mean for G?



- Critical groups agree [Corri-Rossin]! One can check that

$$\kappa(G^*) \cong \kappa(G)$$

- We now fire *circuits*, passing chips according to a rule determined by how these circuits intersect (depends on an *orientation*)

Generalizing to all graphs

- What happens if G is not planar? We no longer have a 'dual graph' but can we still perform 'dual chip-firing'?
- Yes! Think in terms of matrices and lattices (and use some intuition from matroids/Gale duality).
- The reduced Laplacian $\tilde{\mathcal{L}}(G)$ can be computed as

$$\tilde{\mathcal{L}}(G) = \tilde{\partial}\tilde{\partial}^T$$

  where $\tilde{\partial}$ is the (reduced) *incidence matrix* of the graph G.
- The rows of $\tilde{\partial}$ form an integer basis for its row space (called the *lattice of integral cuts* of G).
- Think of classical chip firing on the set of *edges* incident to the nonroot vertices (often called *vertex cuts*).

Dualizing

- Classical chip-firing is determined by a (certain) integer basis for the *row space* of $\tilde{\partial}$.
- For the *dual* picture we seek an integer basis for the *kernel* of $\tilde{\partial}$ (called the *lattice of integral flows* of G).
- For any such basis $\iota = (\mathbf{f}_1 \ldots \mathbf{f}_g)$ let

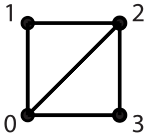
$$\mathcal{L}^*(G) = \iota^T \iota$$

denote the *dual Laplacian* (w/ respect to this choice of basis).

Proposition (essentially Bacher, De La Harpe, Nagnibeda)

For any graph G we have

$$\kappa(G) \cong \mathbb{Z}^g / im\mathcal{L}^*.$$



$$\tilde{\partial} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \tilde{\mathcal{L}} = \tilde{\partial}\tilde{\partial}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

ker$\tilde{\partial}$ has basis $\iota = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ so that $\mathcal{L}^*(G) = \iota^T \iota = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Cycle chip-firing

- After fixing a basis $\iota = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$, the matrix $\mathcal{L}^*$ determines a potential 'chip-firing' rule for these elements.

Definition

For a graph G we say a basis $\iota = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ is a cycle M-basis if the matrix $\mathcal{L}^*$ is an avalanche finite redistribution matrix.

- If G admits a cycle M-basis then we have a good notion of 'chip-firing' on the elements of ι, with notions of z-superstable configurations etc.

Easy cases

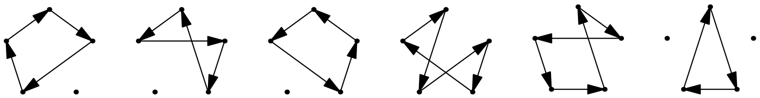
Proposition

If G is a planar graph then G admits a cycle M-basis

- Embed G in the plane, orient the bounded regions consistently, and take the corresponding basis vectors.

Proposition

Both K_5 and $K_{3,3}$ admit cycle M-bases.



$$\mathcal{L}^* = \begin{bmatrix} 4 & -1 & -2 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & -1 \\ -2 & 0 & 4 & -3 & -1 & 0 \\ 0 & 0 & -3 & 5 & 0 & -1 \\ 0 & 0 & -1 & 0 & 5 & -2 \\ 0 & -1 & 0 & -1 & -2 & 3 \end{bmatrix}$$

Main result

Theorem

Any graph G admits a cycle M-basis.

- FACT: if L is a square matrix with $L_{ij} \leq 0$ for all $i \neq j$ then L is avalanche finite $\Leftrightarrow$ real part eigenvalues of L are positive (in which case L is a called a *(non-singular) M-matrix*).
- Recall that $\mathcal{L}^* = \iota^T \iota$ and hence $\mathcal{L}^*$ is positive definite (recall that $\mathcal{L}^*$ is invertible.
- From this it follows that $\mathcal{L}^*$ will have positive real eigenvalues. Hence enough to show that the off-diagonals are nonpositive.

Theorem

Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_g\}$ is an integral basis for a lattice $\Lambda \subset \mathbb{R}^d$. Then there exists an integral basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ for Λ with the property that $\mathbf{f}_i \cdot \mathbf{f}_j \leq 0$ for all $i \neq j$.

z-superstables and more things in bijection with $\tau(G)$

From [GK] we know each equivalence class contains a unique 'energy-minimizing' z-configuration.

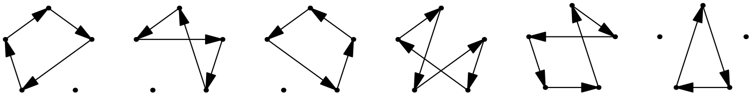
Proposition

Suppose G has a cycle M-basis with associated dual Laplacian $\mathcal{L}^*$. Then the number of z-superstable configurations of G is given by $|\tau(G)|$, the number of spanning trees of G.

- In the classical case the superstable configs have a lot to say about the graph (external activity, Tutte polynomial, etc.)
- Interpretation here not so clear.

Example

- For the case of the graph K_5 (and our example of $\mathcal{L}^*$ from above) we get 125 elements that form a 'multicomplex'



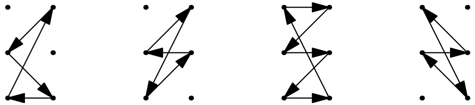
- The maximal elements are $\{000112, 000211, 010022, 010210, 010300, 020021, 020040, 020111, 021020, 021110, 030101, 100102, 101020, 101110, 130020, 130110, 200021, 200111, 210020, 210110, 300020, 310000\}$.
- The degree sequence is given by $\mathbf{c} = (1, 6, 19, 38, 39, 19, 3)$, where the number of z-superstable configurations of degree $d-1$ is given by the entry c_d.

Circuit M-bases

- Although any graph admits a cycle M-basis the elements in our basis for ker $\tilde{\partial}$ can have lots of large integers.
- Want to find an M-basis consisting only of *circuits*: each entry should be $0, -1, 1$, with the the nonzero entries corresponding to some circuit (simple closed path) of G.
- Such a basis will be called a *circuit M-basis*.

Proposition

Planar graphs as well as the graphs K_5 and $K_{3,3}$ admit circuit M-bases.



Further thoughts and open questions

- Main open question: Does any graph admit *circuit* M-basis? Perhaps for a class of graphs?
- Restrictions on the size of circuits in a circuit M-basis? (partial results)
- Find an explicit bijection between the set of z-superstable and the set $\tau(G)$ of spanning trees.
- What do the z-superstable configurations count?
  - In the case of classical chip-firing, the number of superstable configs of degree d are given by the number of spanning trees with d *externally passive* edges.
  - Notion of *activity* here?
- More general unimodular matroids?