Cycle and circuit chip-firing on graphs

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Chip-firing on graphs

- The classical theory of chip-firing involves a simple game played on the vertices of a graph, with connections to
  - self-organized criticality in statistical physics.
  - underlying algebraic structure.
  - divisor theory in algebraic geometry.

- Recent books:

- For us: Chip-firing on a graph $G$ says something about $\mathcal{M}(G)^*$, the dual matroid of the underlying graphic matroid. What about dual chip-firing?
Suppose $G$ is a finite simple graph on vertex set $\{0, 1, \ldots, n\}$ with specified root vertex 0, and edge set $E$.

- A configuration of chips is a vector $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$.
- A vertex $i$ can fire when $c_i \geq \deg(u)$.
- The vertex passes chips to each of its neighbors (one for each edge connecting it to $i$), resulting in a new configuration $c'$.
- A configuration is stable if no vertex can fire.
- The root can fire only when the configuration is stable, passing a chip to each of its neighbors.
Example
FACT: If $G$ is connected all firings stabilize. Why?

Study the dynamics - what configurations do we see ‘many times’?

Given any configuration there is a unique ‘recurrent’ configuration that one can obtain via stabilization.

The set of ‘critical configurations’ form an abelian group $\kappa(G)$ called the ‘critical group’ of $G$. 
Superstable configurations

- We say a configuration superstable if no set of nonroot vertices can fire simultaneously.

In the above graph, 202 is stable but not superstable. Superstable configurations are:
{000, 100, 010, 001, 200, 002, 101, 111}

- Set of superstable configurations are in a simple bijection with the set of critical configurations.

- Superstable configurations have connections to $G$-parking functions, Tutte polynomials (Merino), $h$-vectors, etc.
Some linear algebra

- The dynamics of chip-firing can be encoded in the (reduced) Laplacian of $G$, an $n \times n$ symmetric matrix that encodes $G$.

\[
\tilde{L}(G) = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 2
\end{bmatrix}
\]

- Firing a vertex $i$ corresponds to subtracting the $i$th row of $\tilde{L}(G)$. Two configurations $c$ and $d$ are equivalent if $d - c \in \text{im}\tilde{L}(G)$.

- Recovers the critical group as

\[
\kappa(G) \cong \mathbb{Z}^n / \text{im}\tilde{L}(G)
\]

- As a corollary we see that $|\kappa(G)|$ is given by $\det \tilde{L}(G) = \text{the number of spanning trees of } G$. 
Generalized chip-firing

Gabrielov/Dhar generalized this idea to *abelian avalanche models*.  

- Consider an $n \times n$ *redistribution matrix* $\Delta$ where

  \[
  \Delta_{ii} > 0 \text{ for all } i; \quad \Delta_{ij} \leq 0 \text{ for all } i \neq j.
  \]

- A vector $h \in \mathbb{Z}^n$ defines a configuration, and a site $i$ can fire if $h_i \geq \Delta_{ii}$.

- In this case replace $h$ with $h - \Delta^T e_i$.

- Notions of *stable* configurations are similar.

- Not all matrices give good firing rules - $\Delta$ is said to be *avalanche finite* if any initial configuration and firing sequence will eventually terminate in a stable configuration.
Generalized chip-firing

Guzmán and Klivans have developed a theory of chip-firing for any avalanche finite redistribution matrix $L$.

- Two configurations $f$ and $g$ are equivalent if $g - f \in \text{im} L$.
- A configuration is $z$-superstable if all entries are nonnegative and no multiset of sites can fire simultaneously.
- GK prove that every equivalence class contains a unique $z$-superstable configuration.
- In fact $f$ is $z$-superstable if and only if $f$ is the unique minimizer of

$$\min_{g \sim f, g \geq 0} E(g),$$

where for any configuration $f$ define its energy to be

$$E(f) = f^T L f.$$
Back to graphs: planarity and dual chip-firing

- If $G$ is a planar graph (with an embedding) what does chip-firing on the dual graph $G^*$ mean for $G$?

- Critical groups agree [Corri-Rossin]! One can check that

$$\kappa(G^*) \cong \kappa(G)$$

- We now fire circuits, passing chips according to a rule determined by how these circuits intersect (depends on an orientation)
Generalizing to all graphs

▶ What happens if $G$ is not planar? We no longer have a ‘dual graph’ but can we still perform ‘dual chip-firing’?

▶ Yes! Think in terms of matrices and lattices (and use some intuition from matroids/Gale duality).

▶ The reduced Laplacian $\tilde{\mathcal{L}}(G)$ can be computed as

$$\tilde{\mathcal{L}}(G) = \tilde{\partial} \tilde{\partial}^T$$

where $\tilde{\partial}$ is the (reduced) incidence matrix of the graph $G$.

▶ The rows of $\tilde{\partial}$ form an integer basis for its row space (called the lattice of integral cuts of $G$).

▶ Think of classical chip firing on the set of edges incident to the nonroot vertices (often called vertex cuts).
Dualizing

- Classical chip-firing is determined by a (certain) integer basis for the row space of $\tilde{\partial}$.

- For the dual picture we seek an integer basis for the kernel of $\tilde{\partial}$ (called the lattice of integral flows of $G$).

- For any such basis $\iota = (f_1 \ldots f_g)$ let

$$\mathcal{L}^*(G) = \iota^T \iota$$

denote the dual Laplacian (w/ respect to this choice of basis).

**Proposition (essentially Bacher, De La Harpe, Nagnibeda)**

*For any graph $G$ we have*

$$\kappa(G) \cong \mathbb{Z}^g / \text{im}\mathcal{L}^*.$$
\[ \tilde{\partial} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \tilde{\mathcal{C}} = \tilde{\partial} \tilde{\partial}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \]

\[ \ker \tilde{\partial} \text{ has basis } \iota = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \] so that \[ \mathcal{L}^*(G) = \iota^T \iota = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \]
After fixing a basis $\iota = \{f_1, \ldots, f_g\}$, the matrix $L^*$ determines a potential ‘chip-firing’ rule for these elements.

**Definition**

For a graph $G$ we say a basis $\iota = \{f_1, \ldots, f_g\}$ is a cycle $M$-basis if the matrix $L^*$ is an avalanche finite redistribution matrix.

If $G$ admits a cycle $M$-basis then we have a good notion of ‘chip-firing’ on the elements of $\iota$, with notions of $z$-superstable configurations etc.
Easy cases

Proposition

If $G$ is a planar graph then $G$ admits a cycle $M$-basis

- Embed $G$ in the plane, orient the bounded regions consistently, and take the corresponding basis vectors.

Proposition

Both $K_5$ and $K_{3,3}$ admit cycle $M$-bases.

\[
\mathcal{L}^* = \begin{bmatrix}
4 & -1 & -2 & 0 & 0 & 0 \\
-1 & 4 & 0 & 0 & 0 & -1 \\
-2 & 0 & 4 & -3 & -1 & 0 \\
0 & 0 & -3 & 5 & 0 & -1 \\
0 & 0 & -1 & 0 & 5 & -2 \\
0 & -1 & 0 & -1 & -2 & 3 \\
\end{bmatrix}
\]
Main result

Theorem

Any graph $G$ admits a cycle $M$-basis.

- FACT: if $L$ is a square matrix with $L_{ij} \leq 0$ for all $i \neq j$ then $L$ is avalanche finite $\iff$ real part eigenvalues of $L$ are positive (in which case $L$ is a called a (non-singular) $M$-matrix).

- Recall that $L^* = \iota^T \iota$ and hence $L^*$ is positive definite (recall that $L^*$ is invertible).

- From this it follows that $L^*$ will have positive real eigenvalues. Hence enough to show that the off-diagonals are nonpositive.

Theorem

Suppose $\{v_1, \ldots, v_g\}$ is an integral basis for a lattice $\Lambda \subset \mathbb{R}^d$.
Then there exists an integral basis $\{f_1, \ldots, f_g\}$ for $\Lambda$ with the property that $f_i \cdot f_j \leq 0$ for all $i \neq j$. 
$\tau$-superstables and more things in bijection with $\tau(G)$

From [GK] we know each equivalence class contains a unique ‘energy-minimizing’ $\tau$-configuration.

**Proposition**

*Suppose $G$ has a cycle $M$-basis with associated dual Laplacian $\mathcal{L}^*$. Then the number of $\tau$-superstable configurations of $G$ is given by $|\tau(G)|$, the number of spanning trees of $G$.***

- In the classical case the superstable configs have a lot to say about the graph (external activity, Tutte polynomial, etc.)
- Interpretation here not so clear.
Example

For the case of the graph $K_5$ (and our example of $\mathcal{L}^*$ from above) we get 125 elements that form a ‘multicomplex’

The maximal elements are

$$\{000112, 000211, 010022, 010210, 010300, 020021, 020040, 020111, 021020, 021110, 030101, 100102, 101020, 101110, 130020, 130110, 200021, 200111, 210020, 210110, 300020, 310000\}.$$

The degree sequence is given by $c = (1, 6, 19, 38, 39, 19, 3)$, where the number of $z$-superstable configurations of degree $d - 1$ is given by the entry $c_d$. 
Circuit $M$-bases

- Although any graph admits a cycle $M$-basis the elements in our basis for $\ker \tilde{\partial}$ can have lots of large integers.

- Want to find an $M$-basis consisting only of circuits: each entry should be $0, -1, 1$, with the nonzero entries corresponding to some circuit (simple closed path) of $G$.

- Such a basis will be called a circuit $M$-basis.

**Proposition**

*Planar graphs as well as the graphs $K_5$ and $K_{3,3}$ admit circuit $M$-bases.*
Further thoughts and open questions

- Main open question: Does any graph admit circuit $M$-basis? Perhaps for a class of graphs?
- Restrictions on the size of circuits in a circuit $M$-basis? (partial results)
- Find an explicit bijection between the set of $z$-superstable and the set $\tau(G)$ of spanning trees.
- What do the $z$-superstable configurations count?
  - In the case of classical chip-firing, the number of superstable configs of degree $d$ are given by the number of spanning trees with $d$ externally passive edges.
  - Notion of activity here?
- More general unimodular matroids?