A CW-complex of Monotone Polyhedral Paths.

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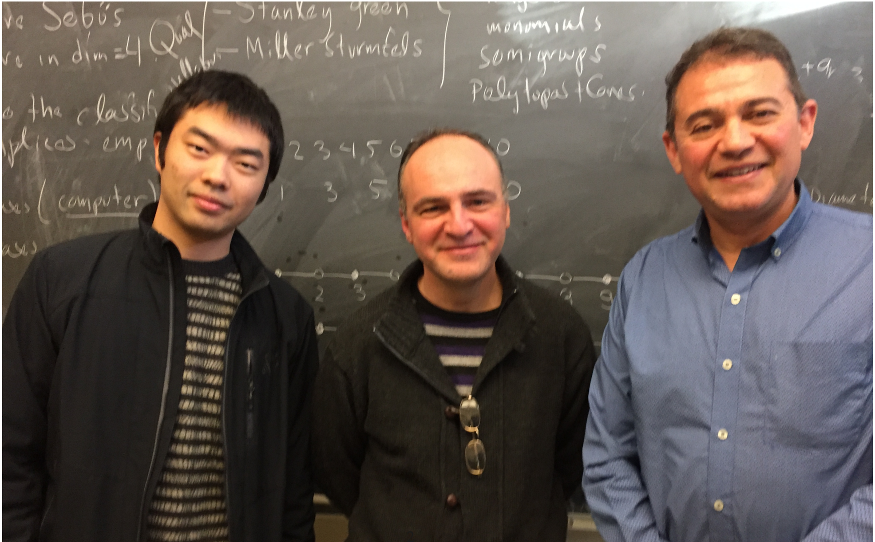
University of California, Davis

AMS Central Sectional Meeting — September of 2020

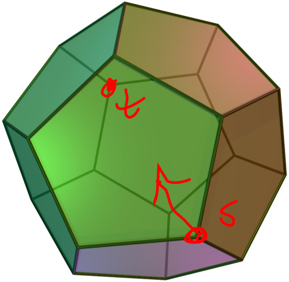
Our Team

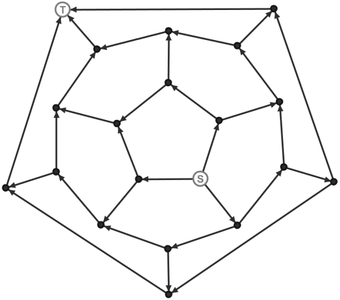
joint work with Christos Athanasiadis and Zhenyang Zhang



MONOTONE PATHS

- Every generic linear functional f induces an orientation on the graph of P. We call this directed graph $\omega(P,f)$.
- Note that $\omega(P,f)$ is acyclic and has a unique source $v_{\min}$ and a unique sink $v_{\max}$.





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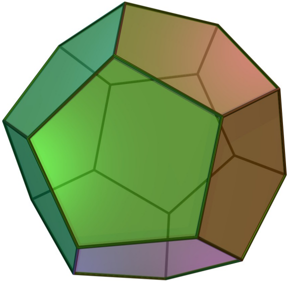

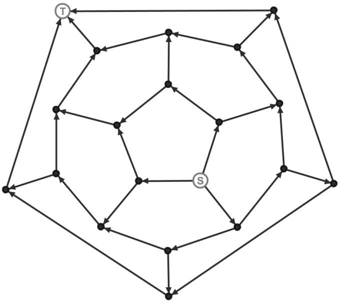

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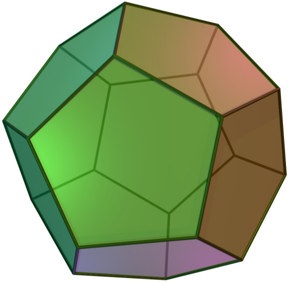
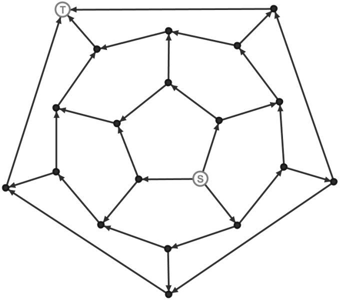


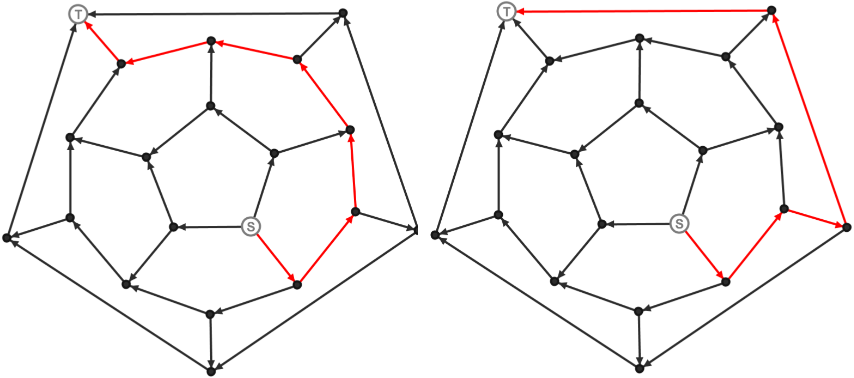


Monotone paths on Polytopes

An *f-monotone path* on P is any directed path in $\omega(P, f)$ from $v_{\min}$ to $v_{\max}$.

$= S$

$= t$



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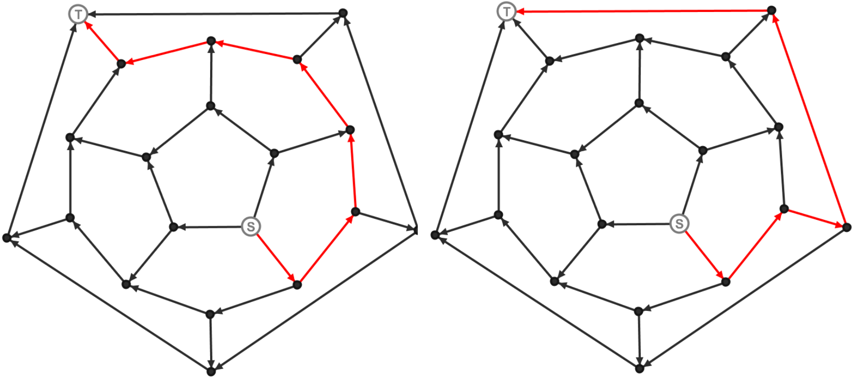
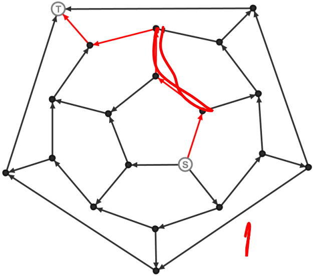


THE FLIP GRAPH OF MONOTONE PATHS

Two f-monotone paths differ by a polygon flip across a 2-dimensional face F if they agree on all edges except follow two different paths on F.



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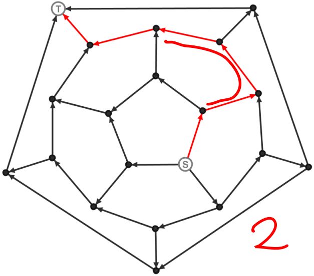

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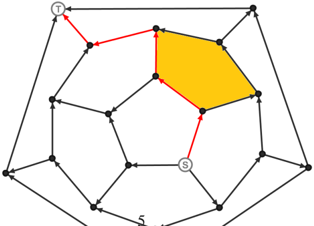
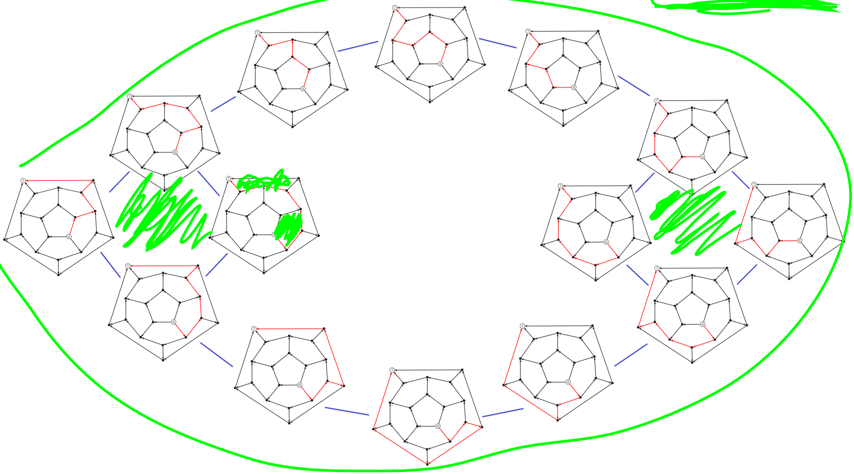


Flip graph of Monotone paths

The flip graph is the (undirected) graph with nodes all f-monotone paths on P and edges are pairs of monotone paths which differ by a polygon flip.

Shadow vertex coherent



Our key questions

Question A

What are the extremal values for **number of monotone paths** for every objective function f on polytopes P with fixed number of vertices and dimension?

Question B

Can we bound the **diameter of polygon flip graph** for polytopes P with fixed number of vertices and dimension?

Answering the questions requires understanding the **SPACE OF ALL MONOTONE PATHS**.

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Answer A: Extreme values for number of monotone paths

Theorem

Let $\mu(P,f)$ be the number of monotone paths on polytope P with objective function f.

- *For all 3-dimensional polytopes P with n vertices,*

$$\left\lceil \frac{n}{2} \right\rceil + 2 \leq \mu(P,f) \leq T_{n-1},$$

  where T_n is the Tribonacci numbers defined by the recurrence $T_0 = T_1 = 1$, $T_2 = 2$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.

- *For all d-dimensional ($d \geq 4$) polytopes P on n vertices,*

$$\left\lceil \frac{dn}{2} \right\rceil + 2 - n \leq \mu(P,f) \leq 2^{n-2}.$$

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Upper bounds for diameter of flip graphs

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Can we bound the diameter of flip graphs for polytopes P with fixed number of vertices and dimension?

Remark: This is a very natural type of question:
Sleator-Tarjan-Thurston investigated the diameter of **associahedra** in terms of triangulation flips.

Theorem

Let $G(P,f)$ be the flip graph of polytope P on objective function f. For any 3-dimensional polytope P on n vertices.

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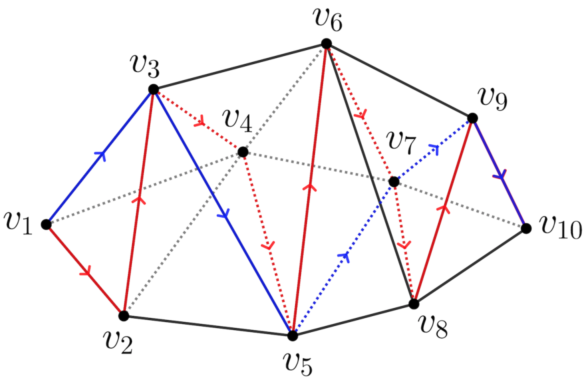

Two paths of distance 16. The flips are $\{1,2,3\}$, $\{7,9,10\}$, $\{7,8,10\}$, $\{8,9,10\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{1,2,3\}$, $\{5,7,8\}$, $\{5,6,8\}$, $\{6,8,9\}$, $\{6,7,9\}$, $\{7,9,10\}$, $\{7,8,10\}$ and $\{8,9,10\}$.

A CW-complex of monotone Paths (Fiber polytopes!)

P d-dim

- **Theorem** (Billera-Kapranov- Sturmfels 1994) There is a CW complex, built from a linear functional on a d-dimensional convex polytope, whose 1-skeleton is the entire flip graph. It has the homotopy type of the $(d-2)$-sphere.
- **Corollary** The polygon flip graph is connected, because CW complex is actually connected
- **Theorem** (Athanasiadis-Edelman-Reiner 2000) Graph of f-monotone paths on a d-polytope P is $(d-1)$-connected for simple polytopes, but the graph is 2-connected for any d-polytope with $d \geq 3$.

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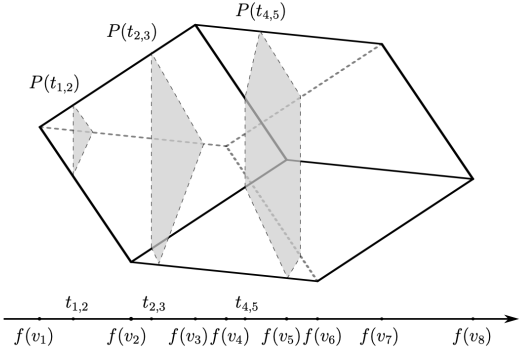
Fiber Polytopes

The function f yields a linear map of P to a line segment. There are finitely many different **fibers**



Theorem (Billera-Sturmfels 1992) The Minkowski sum of all the fibers gives rise to a **fiber polytope**, whose vertices are in bijection with the **coherent** monotone paths. The **coherent** Monotone paths are connected by polygon flips. **Examples:** Secondary Polytopes!!

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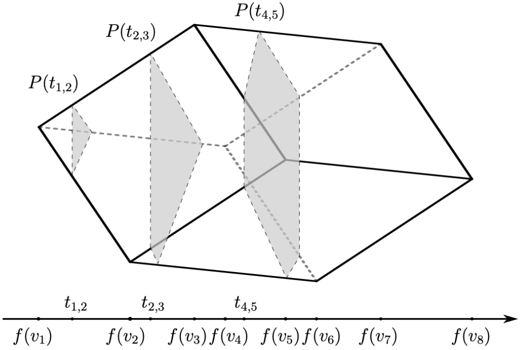

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Open Problems!!

Question

The fiber polytopes we study here are **Monotone Path Polytopes**. Some researchers have studied them. E.g., MPP of a simplex is a cube, the MPP of a zonotope is another zonotope. **Can you characterize other MPPs for famous polytopes?**

YES! Joint work with Alex Black 2020, complete combinatorial characterizations of MPPs of Platonic Solids, Archimedean solids, cross-polytopes of arbitrary dimension. Check Arxiv soon!!

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What is the maximum number of monotone paths on 3-dimensional simple polytopes on $2n$ vertices? Is it $F_{n+2}+1$, where F_n is the Fibonacci numbers, achieved by wedges of (n+1)-gon?

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Gracias!

Merci!
Thank you!

Take care!